The existence and non-existence of ground state solutions to Schrödinger systems with general potentials

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Abstract

In this paper, we first study the existence of ground state solutions for the following Schrödinger systems { - [?] u + V [?] u = G u (u , v) , x [?] R N , - [?] v + V [?] v = G v (u , v) , x [?] R N , u , v > 0 , u , v [?] H 1 (R N) , where N[?]3 and G [?] C 2 ((R +) 2 , R) . And then, by using variational method and projections on Nehari-Poho z aev type manifold, we will prove the nonexistence of ground state solutions for the coupled Schrödinger systems { - [?] u + V (x) u = G u (u , v) , x [?] R N , - [?] v + V (x) v = G v (u , v) , x [?] R N , u , v > 0 , u , v [?] H 1 (R N) .

The existence and non-existence of ground state solutions to Schrödinger systems with general potentials *

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Abstract: In this paper, we first study the existence of ground state solutions for the following Schrödinger systems

$$\begin{cases} -\Delta u + V_\infty u = G_u(u,v), & x\in\mathbb{R}^N, \\ -\Delta v + V_\infty v = G_v(u,v), & x\in\mathbb{R}^N, \\ u,\ v>0, u,\ v\in H^1(\mathbb{R}^N), \end{cases}$$

where $N\geq 3$ and $G\in C^2\left((\mathbb{R}^+)^2,\mathbb{R}\right)$. And then, by using variational method and projections on Nehari-Pohožaev type manifold, we will prove the nonexistence of ground state solutions for the coupled Schrödinger systems

$$\begin{cases} -\Delta u + V(x)u = G_u(u,v), & x\in\mathbb{R}^N, \\ -\Delta v + V(x)v = G_v(u,v), & x\in\mathbb{R}^N, \\ u,\ v>0, u,\ v\in H^1(\mathbb{R}^N). \end{cases}$$

Keywords: Coupled Schrödinger systems; Nehari-Pohožaev type manifold; Ground state solution; Variational method

1 Introduction and main results

In this paper, we are concerned with the following elliptic system

$$\begin{cases} -\Delta u + V(x)u = G_u(u,v), & x\in\mathbb{R}^N, \\ -\Delta v + V(x)v = G_v(u,v), & x\in\mathbb{R}^N, \\ u,\ v>0, u,\ v\in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N\geq 3$ and the potential function V satisfies:

(V_1) $V\in C^1(\mathbb{R}^N,\mathbb{R})$ with $V_0:=\inf_{x\in\mathbb{R}^N}V(x)>0$;

(V_2) $V_\infty:=\lim_{|x|\to\infty}V(x)<\infty$;

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(V_3) $\quad t \mapsto V(t^2x) + \frac{1}{N+1}(\nabla V(t^2x), t^2x)$ is decreasing on $(0, \infty)$ for all $x \in \mathbb{R}^N$.

In order to obtain our results, we will consider the following function space

$$\mathcal{S} := \left\{ \begin{array}{c} f \in C^1(\mathbb{R}^+,\ \mathbb{R}^+), f(0) = 0 \text{ and } f'(s) \geq 0 \text{ for } s > 0; \\ \text{there exist constants } \ p \in (2, 2^*) \text{ and } C_0 > 0 \text{ such that} \\ 0 < f(s) \leq C_0(1 + s^{p-2}) \text{ for all } s > 0. \end{array} \right\},$$

where $2^* = \frac{2N}{N-2}$ $(N \geq 3)$ and we make following hypothesis on $G : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$:

(G) $\quad$ there exist three functions $f_i \in \mathcal{S}$, $i = 1, 2, 3$, such that

$$G(u, v) = \int_0^{\sqrt{uv}} f_1(s)sds + \int_0^u f_2(s)sds + \int_0^v f_3(s)sds \text{ for any } (u, v) \in \mathbb{R}^+ \times \mathbb{R}^+$$

with $f_2(s) + f_3(s) \to \infty$ as $s \to \infty$.

System (1.1) originates from the following system of nonlinear Schrödinger equations:

$$\begin{cases} -i\dfrac{\partial \Psi}{\partial t} = \Delta\Psi - V(x)\Psi + G_1(\Psi), \quad x \in \mathbb{R}^N, \quad t \geq 0, \\ -i\dfrac{\partial \Phi}{\partial t} = \Delta\Phi - V(x)\Phi + G_2(\Phi), \quad x \in \mathbb{R}^N, \quad t \geq 0, \end{cases}$$

where i denotes the imaginary unit, V is the relevant potentials, Ψ and Φ represent the condensate wave functions. Systems of this type appears in the studies of nonlinear optics, Bose-Einstein condensation, Hartree-Fock theory for a double condensate, gap solitons in photonic crystals and so on, and we refer the readers to [4–6, 19]. Recently, the study of this problem have received more and more attention from the mathematical community.

In 2016, Manassés and João [17] proved the existence of nontrivial solution for the following elliptic systems:

$$\begin{cases} -\Delta u + V(x)u = g(x, v), \quad \text{in } \ \mathbb{R}^2, \\ -\Delta v + V(x)v = f(x, u), \quad \text{in } \ \mathbb{R}^2, \\ u, v > 0, \ u, v \in H^1(\mathbb{R}^2), \end{cases} \tag{1.2}$$

where $V : \mathbb{R}^2 \to \mathbb{R}$ may change sign and vanish, f and g are superlinear at infinity and have exponential subcritical or critical growth of the Trudinger-Moser type.

Very recently, Chen and Li [2] studied the following Schrödinger systems:

$$\begin{cases} -\Delta u + V_1(x)u = F_u(x, u, v), \quad x \in \mathbb{R}^N, \\ -\Delta v + V_2(x)v = F_v(x, u, v), \quad x \in \mathbb{R}^N, \\ u, \ v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.3}$$

where V_1, V_2 are periodic in x, the nonlinearity F is local super-quadratic. By using approximation argument and variational method, they proved the existence of Nehari-Pankov type ground state solution and the least energy solutions of system (1.3).

After that, by using topological arguments and sharp exponential decay estimates, Liliane et al [12] investigated the existence of bound state solution for the following elliptic system:

$$\begin{cases} -\Delta u+\mu_1 u=(1+a(x))\dfrac{u(u^2+v^2)}{1+s(u^2+v^2)}+\lambda v \quad \text{in } \ \mathbb{R}^N, \\ -\Delta v+\mu_2 v=(1+a(x))\dfrac{v(u^2+v^2)}{1+s(u^2+v^2)}+\lambda u \quad \text{in } \ \mathbb{R}^N, \\ u,\ v\in H^1(\mathbb{R}^N). \end{cases} \tag{1.4}$$

Qin et al. [13] obtained a positive bound state solution for the following Schrödinger systems:

$$\begin{cases} -\Delta u+\lambda_1 u=a(x)F_u(u,v), \quad x\in\mathbb{R}^N, \\ -\Delta v+\lambda_2 v=a(x)F_v(u,v), \quad x\in\mathbb{R}^N, \\ u,\ v\in H^1(\mathbb{R}^N), \end{cases} \tag{1.5}$$

where $F\in C^2((\mathbb{R}^+)^2,\mathbb{R})$, $a\in C^2(\mathbb{R}^N,\mathbb{R})$ satisfies the following assumptions:

(a_1) $\nabla a(x)\cdot x\geq 0$, $a(x)+\frac{\nabla a(x)\cdot x}{N}<a_\infty:=\lim_{|x|\to\infty}a(x)$ for all $x\in\mathbb{R}^N$;

(a_2) $\nabla a(x)\cdot x+\frac{x\cdot H(x)\cdot x}{N}\geq 0$ for all $x\in\mathbb{R}^N$, where H denotes the Hessian matrix of a. Moreover, they also studied the nonexistence result for a minimizing problem.

In addition, there are many results about semiclassical state solutions, sign-changing solutions, normalized solutions and ground state solutions of elliptic systems, see for example, [1,7,8,10,11,15,20,22–29] and the references therein.

Inspired by above works, in this paper, our goal is to investigate the existence and nonexistence of ground state solutions by using variational method. We will constrain the functional related with system (1.1) to the so-called Nehari-Pohožaev type manifold

$$\mathcal{M}=\left\{z\in E:\left\langle I'(z),z\right\rangle+2\mathcal{P}(z)=0\right\},$$

which is given in (2.5) below. However, we can't prove that $\mathcal{M}$ is a natural constraint due to the conditions of V. So we don't get a bound state solution of system (1.1).

Before stating our main results, let $H^1(\mathbb{R}^N)$ denote the usual Sobolev space, by (V_2) and (V_3), we use an equivalent norm on $H^1(\mathbb{R}^N)$:

$$\|u\|=\left(\int_{\mathbb{R}^N}(|\nabla u|^2+V(x)u^2)dx\right)^{\frac{1}{2}}, \ \ \forall\ u\in H^1(\mathbb{R}^N).$$

Let $E:=H^1(\mathbb{R}^N)\times H^1(\mathbb{R}^N)$ be the Hilbert space with the inner product and norm

$$\langle z,\xi\rangle=\langle u,\varphi\rangle+\langle v,\psi\rangle, \qquad \forall\ z=(u,v), \quad \xi=(\varphi,\psi)\in E$$

and

$$\|z\|=\langle z,z\rangle^{\frac{1}{2}}=\left(\|u\|^2+\|v\|^2\right)^{\frac{1}{2}}, \ \ \forall\ z=(u,v)\in E.$$

Throughout the paper, we make use of the following notations:

- $L^s(\mathbb{R}^N)(1\leq s<\infty)$ denotes the Lebesgue space with the norm $|u|_s=\left(\int_{\mathbb{R}^N}|u|^s\,dx\right)^{\frac{1}{s}}$;

- Let $L^s(\mathbb{R}^N, \mathbb{R}^2) := L^s(\mathbb{R}^N) \times L^s(\mathbb{R}^N)$ with the norm:

$$\|z\|_s = (|u|_s^s + |v|_s^s)^{\frac{1}{s}} \text{ for any } z = (u, v) \in L^s(\mathbb{R}^N, \mathbb{R}^2);$$

- For any $(s, t) \in \mathbb{R}^2$, $\nabla G(s, t) := (G_s(s, t), G_t(s, t))$;
- For any $z = (u, v) \in E$, let $z_t(x) := tz(t^{-2}x) = (tu(t^{-2}x), tv(t^{-2}x))$ for $t > 0$;
- For all $x \in \mathbb{R}^N$ and $r > 0$, $B_r(x) := \{y \in \mathbb{R}^N : |y - x| < r\}$;
- $C_1, C_2, C', \ldots$ denote positive constants possibly different in different space.

From Sobolev embedding theorems, it is easy to see that the embedding $E \hookrightarrow L^s(\mathbb{R}^N, \mathbb{R}^2)$ is continuous for all $2 \leq s \leq 2^*$ and $E \hookrightarrow L^s_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ is compact for all $2 \leq s < 2^*$. So there exists a constant $\eta_s > 0$ such that

$$\|z\|_s \leq \eta_s \|z\|, \quad \forall\, z \in E, \quad s \in [2, 2^*]. \tag{1.6}$$

From a variational point of view, weak solutions of system (1.1) correspond to critical points of the following C^1-functional on E:

$$I(z) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla z|^2 + V(x)\,|z|^2)dx - \int_{\mathbb{R}^N} G(z)dx. \tag{1.7}$$

Moreover, for all $z,\ \zeta \in E$, we have

$$\langle I'(z), \zeta \rangle = \langle z, \zeta \rangle - \int_{\mathbb{R}^N} \nabla G(z) \cdot \zeta dx. \tag{1.8}$$

The limit form of system (1.1) is the following Schrödinger systems

$$\begin{cases} -\Delta u + V_\infty u = G_u(u, v), & x \in \mathbb{R}^N, \\ -\Delta v + V_\infty v = G_v(u, v), & x \in \mathbb{R}^N, \\ u,\ v > 0, u,\ v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.9}$$

and its energy functional is defined in E as follows:

$$I_\infty(z) = \frac{1}{2}\int_{\mathbb{R}^N} (|\nabla z|^2 + V_\infty\,|z|^2)dx - \int_{\mathbb{R}^N} G(z)dx. \tag{1.10}$$

Under (G), our first goal in this paper is to give a concise proof of the ground state solution of system (1.9). We borrow the ideas from Tang and Chen in [3]. Now we are ready to state the main results of this paper:

Theorem 1.1. Assume that G satisfies (G). Then system (1.9) has a solution $\bar{z} \in E \setminus \{0\}$ such that $I_\infty(\bar{z}) = \inf_{\mathcal{M}_\infty} I_\infty$, where $\mathcal{M}_\infty$ is given in (2.6).

Theorem 1.2. Assume that V and G satisfy $(V_1) - (V_3)$ and (G). Then $m := \inf_{z \in \mathcal{M}} I(z)$ given in (2.7) below is not a critical level for the functional I.

Remark 1.3. By a simple calculation, (V_3) implies the following inequality:

$$(N+1)t^{2N+2}\big(V(x)-V(t^2x)\big)-(1-t^{2N+2})(\nabla V(x),x)>0,\quad \forall\, t>0,\quad x\in\mathbb{R}^N. \tag{1.11}$$

Moreover, jointly with (V_2), we have

$$V(x)+\frac{(\nabla V(x),x)}{N+1}>V_\infty,\quad \forall\, x\in\mathbb{R}^N \tag{1.12}$$

and

$$(\nabla V(x),x)\le 0,\quad \forall\, x\in\mathbb{R}^N. \tag{1.13}$$

From $(V_1)-(V_3)$, we can see that $V_\infty<V(x)\le\sup_{x\in\mathbb{R}^3}V(x)<\infty$ and $(\nabla V(x),x)\to 0$ as $|x|\to\infty$. Furthermore, there are indeed functions which satisfy $(V_1)-(V_3)$, an example is given by $V(x)=V_\infty+\frac{1}{1+|x|^{\frac{3}{2}}}$.

Remark 1.4. Condition (G) implies that system (1.1) is a full coupling system, in other words, it cannot be reduced to two independent equations. Moreover, for any $\varepsilon>0$, there exists $C_\varepsilon>0$ such that for all $s\in[p,2^*]$,

$$|G(z)|\le\varepsilon\,|z|^2+C_\varepsilon\,|z|^s\,,\quad |\nabla G(z)|\le\varepsilon\,|z|+C_\varepsilon\,|z|^{s-1}\ \text{ for all } z\in\mathbb{R}^2. \tag{1.14}$$

Furthermore, for any $z\in\mathbb{R}^2\setminus\{0\}$, we can see that $\frac{2NG(tz)+\nabla G(tz)\cdot tz}{t^2}$, $\frac{1}{2}\nabla G(tz)\cdot tz-G(tz)$ are increasing on $t\in(0,+\infty)$.

2 Notations and preliminaries

In this section, we will give some notations and preliminaries, including lemmas that are required in proving the main results.

First, we define the two Pohožaev functionals on E associated with system (1.1) and (1.9):

$$\begin{aligned}\mathcal{P}(z):=&\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla z|^2\,dx+\frac{N}{2}\int_{\mathbb{R}^N}V(x)\,|z|^2\,dx\\&+\frac{1}{2}\int_{\mathbb{R}^N}(\nabla V(x),x)\,|z|^2\,dx-N\int_{\mathbb{R}^N}G(z)dx,\end{aligned} \tag{2.1}$$

and

$$\mathcal{P}_\infty(z):=\frac{N-2}{2}\int_{\mathbb{R}^N}|\nabla z|^2\,dx+\frac{N}{2}\int_{\mathbb{R}^N}V_\infty\,|z|^2\,dx-N\int_{\mathbb{R}^N}G(z)dx. \tag{2.2}$$

Moreover, we define the two Nehari-Pohožaev functionals on E as follows:

$$\begin{aligned}J(z)=&\big\langle I'(z),z\big\rangle+2\mathcal{P}(z)\\=&(N-1)\int_{\mathbb{R}^N}|\nabla z|^2\,dx+(N+1)\int_{\mathbb{R}^N}\left(V(x)+\frac{(\nabla V(x),x)}{N+1}\right)|z|^2\,dx\\&-\int_{\mathbb{R}^N}(2NG(z)+\nabla G(z)\cdot z)dx,\end{aligned} \tag{2.3}$$

and

$$
\begin{aligned}
J_\infty(z) =& \left\langle {I_\infty}'(z), z \right\rangle + 2\mathcal{P}_\infty(z) \\
=&(N-1)\int_{\mathbb{R}^N} |\nabla z|^2 \, dx + (N+1)\int_{\mathbb{R}^N} V_\infty \, |z|^2 \, dx \\
&- \int_{\mathbb{R}^N} (2NG(z) + \nabla G(z)\cdot z) dx.
\end{aligned} \tag{2.4}
$$

Let

$$
\mathcal{M} := \{z \in E \setminus \{0\} : J(z) = 0\} \tag{2.5}
$$

and

$$
\mathcal{M}_\infty := \{z \in E \setminus \{0\} : J_\infty(z) = 0\}. \tag{2.6}
$$

Define

$$
m := \inf_{z\in\mathcal{M}} I(z), \tag{2.7}
$$

and

$$
m_\infty := \inf_{z\in\mathcal{M}_\infty} I_\infty(z). \tag{2.8}
$$

Lemma 2.1. If (G) holds, we have

$$
\frac{1-t^{2N+2}}{2(N+1)}\nabla G(\tau)\cdot\tau - \frac{1+Nt^{2N+2}}{N+1}G(\tau) + t^{2N}G(t\tau) \ge 0, \quad \forall\, t \ge 0, \quad \tau \in \mathbb{R}^2. \tag{2.9}
$$

Proof. It is easy to see that (2.9) holds for $\tau = (0,0)$. For $\tau \neq (0,0)$, let

$$
F(t) = \frac{1-t^{2N+2}}{2(N+1)}\nabla G(\tau)\cdot\tau - \frac{1+Nt^{2N+2}}{N+1}G(\tau) + t^{2N}G(t\tau), \quad t \ge 0. \tag{2.10}
$$

Therefore, we have

$$
F'(t) = t^{2N+1}|\tau|^2\left(\frac{\nabla G(t\tau)\cdot t\tau + 2NG(t\tau)}{|t\tau|^2} - \frac{\nabla G(\tau)\cdot\tau + 2NG(\tau)}{|\tau|^2}\right).
$$

By (G), we have $F'(t) \ge 0$ for $t \ge 1$ and $F'(t) \le 0$ for $0 < t < 1$. Hence, we obtain $F(t) \ge F(1) = 0$, which implies (2.9). $\square$

Moreover, it is easy to see the following inequality holds:

$$
g(t) := 2 - (N+1)t^{2N-2} + (N-1)t^{2N+2} > g(1) = 0, \quad \forall\, t \in [0,1) \cup (1,+\infty). \tag{2.11}
$$

For any $z = (u,v) \in E \setminus \{0\}$ and $t > 0$, recall that $z_t(x) = tz(t^{-2}x) = (tu(t^{-2}x), tv(t^{-2}x))$, from (1.7) and (2.3), we have

$$
I(z_t) = \frac{t^{2N-2}}{2}\int_{\mathbb{R}^N} |\nabla z|^2 \, dx + \frac{t^{2N+2}}{2}\int_{\mathbb{R}^N} V(t^2x)\, |z|^2 \, dx - t^{2N}\int_{\mathbb{R}^N} G(tz) dx, \tag{2.12}
$$

and

$$
\begin{aligned}
J(z_t) =&(N-1)t^{2N-2}\, \|\nabla z\|_2^2 + (N+1)t^{2N+2}\int_{\mathbb{R}^N}\left(V(t^2x) + \frac{(\nabla V(t^2x), t^2x)}{N+1}\right)|z|^2 \, dx \\
&- t^{2N}\int_{\mathbb{R}^N} (2NG(tz) + \nabla G(tz)\cdot tz)\, dx.
\end{aligned} \tag{2.13}
$$

Lemma 2.2. Assume that $(V_1), (V_3)$ and (G) hold. For any $z \in E \setminus \{0\}$ and for all $t > 0$, we have

$$I(z) \geq I(z_t) + \frac{1-t^{2N+2}}{2N+2} J(z) + \frac{2-(N+1)t^{2N-2}+(N-1)t^{2N+2}}{2N+2} \|\nabla z\|_2^2 . \tag{2.14}$$

Proof. From Lemma 2.1, (1.11) and (2.12), we have

$$\begin{aligned}
& I(z) - I(z_t) \\
=& \frac{1-t^{2N-2}}{2} \|\nabla u\|_2^2 + \frac{1}{2}\int_{\mathbb{R}^N} \left(V(x) - t^{2N+2}V(t^2x)\right)|z|^2\,dx \\
& + \int_{\mathbb{R}^N} \left(t^{2N}G(tz) - G(z)\right) dx \\
=& \frac{1-t^{2N+2}}{2N+2} J(z) + \frac{2-(N+1)t^{2N-2}+(N-1)t^{2N+2}}{2N+2} \|\nabla z\|_2^2 \\
& + \frac{1}{2N+2}\int_{\mathbb{R}^N} \left\{(N+1)t^{2N+2}\left(V(x)-V(t^2x)\right) - (1-t^{2N+2})(\nabla V(x), x)\right\}|z|^2\,dx \\
& + \int_{\mathbb{R}^N} \left(\frac{1-t^{2N+2}}{2N+2}\nabla G(z)\cdot z - \frac{1+Nt^{2N+2}}{N+1}G(z) + t^{2N}G(tz)\right) dx \\
\geq& \frac{1-t^{2N+2}}{2N+2} J(z) + \frac{2-(N+1)t^{2N-2}+(N-1)t^{2N+2}}{2N+2} \|\nabla z\|_2^2 . \quad \square
\end{aligned}$$

By Lemma 2.2, we have the following corollaries.

Corollary 2.3. Assume that (G) holds. Then for all $z \in E \setminus \{0\}$ and $t > 0$, we have

$$I_\infty(z) \geq I_\infty(z_t) + \frac{1-t^{2N+2}}{2N+2} J_\infty(z) + \frac{2-(N+1)t^{2N-2}+(N-1)t^{2N+2}}{2N+2} \|\nabla z\|_2^2 . \tag{2.15}$$

Corollary 2.4. Assume that (V_1), (V_3) and (G) hold. Then for all $z \in \mathcal{M}$, we have

$$I(z) = \max_{t>0} I(z_t). \tag{2.16}$$

Lemma 2.5. Assume that $(V_1), (V_2), (V_3)$ and (G) hold. Then for any $z \in E \setminus \{0\}$, there exists a unique $t_z > 0$ such that $z_{t_z} \in \mathcal{M}$.

Proof. Let $z \in E \setminus \{0\}$ be fixed and define a function $h(t) := I(z_t)$ on $\mathbb{R}^+$. By (2.12) and (2.13), we have

$$\begin{aligned}
& h'(t) = 0 \\
\Leftrightarrow& (N-1)t^{2N-3}\|\nabla z\|_2^2 + (N+1)t^{2N+1}\int_{\mathbb{R}^N}\left(V(t^2x) + \frac{(\nabla V(t^2x), t^2x)}{N+1}\right)|z|^2\,dx \\
& - t^{2N-1}\int_{\mathbb{R}^N}\left(2NG(tz) + \nabla G(tz)\cdot tz\right)dx = 0 \\
\Leftrightarrow& J(z_t) = 0 \Leftrightarrow z_t \in \mathcal{M}.
\end{aligned} \tag{2.17}$$

Using $(V_1), (V_2), (G)$ and (2.12), we have

$$\frac{h(t)}{t^{2N+2}} = \frac{1}{2t^4}\int_{\mathbb{R}^N}|\nabla z|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}V(t^2x)|z|^2\,dx - \int_{\mathbb{R}^N}\frac{G(tz)}{|tz|^2}|z|^2\,dx \to -\infty$$

as $t \to +\infty$ and $\lim_{t\to 0} h(t) = 0$, so we have $h(t) > 0$ for $t > 0$ small and $h(t) < 0$ for t large. Therefore $\max_{t\geq 0} h(t)$ is achieved at some $t_z > 0$ such that $h'(t_z) = 0$ and $z_{t_z} \in \mathcal{M}$.

Next we claim that t_z is unique for any $z \in E \setminus \{0\}$. Indeed, for any given $z \in E \setminus \{0\}$, let $t_1, t_2 > 0$ such that $z_{t_1}, z_{t_2} \in \mathcal{M}$, that is, $J(z_{t_1}) = J(z_{t_2}) = 0$. Jointly with (2.14), one has

$$\begin{aligned} I(z_{t_1}) &\geq I(z_{t_2}) + \frac{t_1^{2N+2} - t_2^{2N+2}}{2(N+1)t_1^{2N+2}} J(z_{t_1}) + \frac{2t_1^{2N+2} - (N+1)t_1^2 t_2^{2N-2} + (N-1)t_2^{2N+2}}{2(N+1)t_1^{2N+2}} \|\nabla z\|_2^2 \\ &= I(z_{t_2}) + \frac{2t_1^{2N+2} - (N+1)t_1^2 t_2^{2N-2} + (N-1)t_2^{2N+2}}{2(N+1)t_1^{2N+2}} \|\nabla z\|_2^2 \end{aligned}$$

and

$$\begin{aligned} I(z_{t_2}) &\geq I(z_{t_1}) + \frac{t_2^{2N+2} - t_1^{2N+2}}{2(N+1)t_2^{2N+2}} J(z_{t_2}) + \frac{2t_2^{2N+2} - (N+1)t_2^2 t_1^{2N-2} + (N-1)t_1^{2N+2}}{2(N+1)t_2^{2N+2}} \|\nabla z\|_2^2 \\ &= I(z_{t_1}) + \frac{2t_2^{2N+2} - (N+1)t_2^2 t_1^{2N-2} + (N-1)t_1^{2N+2}}{2(N+1)t_2^{2N+2}} \|\nabla z\|_2^2, \end{aligned}$$

which imply that $t_1 = t_2$. Therefore $t_z > 0$ is unique for any $z \in E \setminus \{0\}$. $\square$

Corollary 2.6. Assume that (G) holds. Then for all $z \in E \setminus \{0\}$, there exists a unique $t_z > 0$ such that $z_{t_z} \in \mathcal{M}_\infty$. Moreover, $I_\infty(z_{t_z}) = \max_{t>0} I_\infty(z_t)$.

Combining Corollary 2.4 with Lemma 2.5, we have the following corollary.

Corollary 2.7. Assume that $(V_1), (V_2), (V_3)$ and (G) hold, we have

$$\inf_{z\in\mathcal{M}} I(z) := m = \inf_{z\in E\setminus\{0\}} \max_{t>0} I(z_t).$$

By a standard argument, we can prove the following lemma.

Lemma 2.8. Assume that (V_1), (V_3) and (G) hold. If $z_n \rightharpoonup \bar{z}$ in E, then

$$I(z_n) = I(\bar{z}) + I(z_n - \bar{z}) + o(1)$$

and

$$J(z_n) = J(\bar{z}) + J(z_n - \bar{z}) + o(1).$$

Next we can formulate the existence of a ground state solution of Nehari-Pohožaev type for system (1.9). Theorem 1.1 is a direct corollary of the following two lemmas.

Lemma 2.9. Assume that (G) holds. Then m_∞ is achieved.

Proof. From (G), we can see for all $z \in E$

$$\begin{aligned} \frac{1}{2}\nabla G(z) \cdot z - G(z) = &\int_0^{\sqrt{uv}} (f_1(\sqrt{uv}) - f_1(s))sds + \int_0^u ((f_2(u) - f_2(s))sds \\ &+ \int_0^v (f_3(v) - f_3(s))sds \geq 0. \end{aligned} \tag{2.18}$$

We introduce a new functional $\Psi_\infty : E \to \mathbb{R}$ as follows:

$$\Psi_\infty(z) = \frac{1}{N+1}\int_{\mathbb{R}^N} |\nabla z|^2\, dx + \frac{1}{2(N+1)}\int_{\mathbb{R}^N} (\nabla G(z)\cdot z - 2G(z))dx, \quad \forall\, z \in E. \tag{2.19}$$

Then $\Psi_\infty(z) = I_\infty(z) \geq m_\infty$ for all $z \in \mathcal{M}_\infty$. Let $\{z_n\} \subset \mathcal{M}_\infty$ be such that $I_\infty(z_n) \to m_\infty$. Since $J_\infty(z_n) = 0$, it follows from (2.18) and (2.19) that

$$m_\infty + o(1) = I_\infty(z_n) \geq \frac{1}{N+1}\|\nabla z_n\|_2^2\,, \tag{2.20}$$

which shows that $\left\{\|\nabla z_n\|_2^2\right\}$ is bounded. On the other hand, from (1.12), (1.14) and $J_\infty(z_n) = 0$, we have for n large,

$$\begin{aligned}(N+1)V_\infty \int_{\mathbb{R}^N} |z_n|^2\, dx &\leq \int_{\mathbb{R}^N} (2NG(z_n) + \nabla G(z_n)\cdot z_n)dx\\ &\leq \varepsilon \int_{\mathbb{R}^N} |z_n|^2\, dx + C_\varepsilon \int_{\mathbb{R}^N} |z_n|^{2^*}\, dx\\ &\leq \varepsilon \int_{\mathbb{R}^N} |z_n|^2\, dx + C_\varepsilon' \left(\int_{\mathbb{R}^N} |\nabla z_n|^2\, dx\right)^{\frac{N}{N-2}},\end{aligned}$$

so if we take $0 < \varepsilon < (N+1)V_\infty$, $\{\|z_n\|_2\}$ is bounded, thus $\{z_n\}$ is bounded in E.

Let

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |z_n|^2\, dx.$$

If $\delta = 0$, by Lions' concentration compactness principle ([21], Lemma 1.21), we have $z_n \to 0$ in $L^s(\mathbb{R}^N, \mathbb{R}^2)$ for $2 < s < 2^*$. Using the fact $J_\infty(z_n) \to 0$, we deduce that $z_n \to 0$ in E, thus $I_\infty(z_n) \to 0$, which contradicts with $I_\infty(z_n) \to m_\infty$. Therefore $\delta > 0$, and then there exists a sequence $\{y_n\} \subset \mathbb{R}^N$ such that

$$\int_{B_1(y_n)} |z_n|^2\, dx > \delta/2.$$

Next we claim that $\{y_n\}$ is unbounded. Indeed, if $\{y_n\}$ is bounded, there exists a $R > 0$ such that $\{y_n\} \subset B_R(0)$. It is clear that $B_1(y_n) \subset B_{R+1}(0)$, and we can see

$$\int_{B_{R+1}(0)} |z_n|^2\, dx \geq \int_{B_1(y_n)} |z_n|^2\, dx > \delta/2,$$

which implies that $z_n \not\to 0$ in $L^2(B_{R+1}(0), \mathbb{R}^2)$. It is impossible, because $z_n \to 0$ in $L^s_{loc}(\mathbb{R}^N, \mathbb{R}^2)$ for $2 \leq s < 2^*$. Hence we may assume that $|y_n| \to \infty$. Let $\hat{z}_n(x) = z_n(x + y_n)$, we have $\|\hat{z}_n\| = \|z_n\|$ and

$$I_\infty(\hat{z}_n) \to m_\infty, \quad J_\infty(\hat{z}_n) \to 0, \quad \int_{B_1(0)} |\hat{z}_n|^2\, dx > \delta/2. \tag{2.21}$$

Therefore, there exists $\bar{z} \in E \setminus \{0\}$ such that, passing to a subsequence if necessary,

$$\begin{cases} \hat{z}_n \rightharpoonup \bar{z} & \text{in } E;\\ \hat{z}_n \to \bar{z} & \text{in } L^s_{loc}(\mathbb{R}^N, \mathbb{R}^2) \text{ for all } s \in [2, 2^*)\,;\\ \hat{z}_n \to \bar{z} & \text{a.e. on } \mathbb{R}^N. \end{cases} \tag{2.22}$$

Let $w_n = \hat{z}_n - \bar{z}$. Then (2.22) and Lemma 2.8 yield

$$\Psi_\infty(\hat{z}_n) = \Psi_\infty(\bar{z}) + \Psi_\infty(w_n) + o(1) \tag{2.23}$$

and

$$J_\infty(\hat{z}_n) = J_\infty(\bar{z}) + J_\infty(w_n) + o(1). \tag{2.24}$$

From (1.10), (2.19), (2.21), (2.23) and (2.24), we have

$$\Psi_\infty(w_n) = m_\infty - \Psi_\infty(\bar{z}) + o(1), \quad J_\infty(w_n) \le -J_\infty(\bar{z}) + o(1). \tag{2.25}$$

If there exists a subsequence $\{w_{n_i}\}$ of $\{w_n\}$ such that $w_{n_i} = 0$, then going to this subsequence, one has

$$I_\infty(\bar{z}) = m_\infty, \quad J_\infty(\bar{z}) = 0. \tag{2.26}$$

Next, we assume that $w_n \neq 0$. We claim $J_\infty(\bar{z}) \le 0$. Otherwise, if $J_\infty(\bar{z}) > 0$, then (2.25) implies $J_\infty(w_n) < 0$ for large n. From Corollary 2.6, there exists $t_n > 0$ such that $(w_n)_{t_n} \in \mathcal{M}_\infty$. From (1.10), (2.11), (2.15), (2.19) and (2.25), one has

$$\begin{aligned}
m_\infty - \Psi_\infty(\bar{z}) + o(1) &= \Psi_\infty(w_n) \\
&= \frac{1}{N+1}\int_{\mathbb{R}^N} |\nabla w_n|^2\, dx + \frac{1}{2N+2}\int_{\mathbb{R}^N} (\nabla G(w_n)\cdot w_n - 2G(w_n))dx \\
&= I_\infty(w_n) - \frac{1}{2N+2} J_\infty(w_n) \\
&\ge I_\infty((w_n)_{t_n}) - \frac{t_n^{2N+2}}{2N+2} J_\infty(w_n) \\
&\ge m_\infty - \frac{t_n^{2N+2}}{2N+2} J_\infty(w_n) \\
&\ge m_\infty,
\end{aligned}$$

which implies that $J_\infty(\bar{z}) \le 0$ due to $\Psi_\infty(\bar{z}) > 0$. In view of Corollary 2.6, there exists $\bar{t} > 0$ such that $\bar{z}_{\bar{t}} \in \mathcal{M}_\infty$. From (1.10), (2.15), (G), (2.21), the weak semicontinuity of norm and Fatou's lemma, we have

$$\begin{aligned}
m_\infty &= \lim_{n\to\infty}\left(I_\infty(\hat{z}_n) - \frac{1}{2N+2} J_\infty(\hat{z}_n)\right) \\
&= \lim_{n\to\infty}\left(\frac{1}{N+1}\int_{\mathbb{R}^N} |\nabla \hat{z}_n|^2\, dx + \frac{1}{2N+2}\int_{\mathbb{R}^N} (\nabla G(\hat{z}_n)\cdot \hat{z}_n - 2G(\hat{z}_n))dx\right) \\
&\ge \frac{1}{N+1}\int_{\mathbb{R}^N} |\nabla \bar{z}|^2\, dx + \frac{1}{2N+2}\int_{\mathbb{R}^N} (\nabla G(\bar{z})\cdot \bar{z} - 2G(\bar{z}))dx \\
&= I_\infty(\bar{z}) - \frac{1}{2N+2} J_\infty(\bar{z}) \\
&\ge I_\infty(\bar{z}_{\bar{t}}) - \frac{\bar{t}^{2N+2}}{2N+2} J_\infty(\bar{z}) \\
&\ge m_\infty - \frac{\bar{t}^{2N+2}}{2N+2} J_\infty(\bar{z}) \ge m_\infty,
\end{aligned}$$

which implies

$$I_\infty(\bar{z}) = m_\infty, \quad J_\infty(\bar{z}) = 0. \quad \square$$

Lemma 2.10. Assume that G satisfies (G). If $\bar{z} \in \mathcal{M}_\infty$ and $I_\infty(\bar{z}) = m_\infty$, then $\bar{z}$ is a critical point of I_∞.

Proof. Assume by contradiction that ${I_\infty}'(\bar{z}) \neq 0$, there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|z - \bar{z}\| \leq 3\delta \quad \Rightarrow \quad \left\|{I_\infty}'(\bar{z})\right\| \geq \varrho. \tag{2.27}$$

First, we prove that

$$\lim_{t \to 1} \|\bar{z}_t - \bar{z}\| = 0, \tag{2.28}$$

where $\bar{z}_t(x) = t\bar{z}(t^{-2}x)$. Arguing by contradiction, we suppose that there exists $\varepsilon_0 > 0$ and a sequence $\{t_n\}$ such that

$$\lim_{n \to \infty} t_n = 1 \quad \text{and} \quad \|\bar{z}_{t_n} - \bar{z}\|^2 \geq \varepsilon_0. \tag{2.29}$$

Since $\bar{z} \in E$, there exist $U \in C_0(\mathbb{R}^N, \mathbb{R}^{2N})$ and $v \in C_0(\mathbb{R}^N, \mathbb{R}^2)$ such that

$$\int_{\mathbb{R}^N} |\nabla \bar{z} - U|^2 \, dx < \frac{\varepsilon_0}{20} \quad \text{and} \quad \int_{\mathbb{R}^N} |\bar{z} - v|^2 \, dx < \frac{\varepsilon_0}{20}. \tag{2.30}$$

By (2.29) and (2.30), we have

$$\begin{aligned}
\|\nabla \bar{z}_{t_n} - \nabla \bar{z}\|_2^2 =& \int_{\mathbb{R}^N} \left|\nabla(t_n \bar{z}(t_n^{-2}x)) - \nabla \bar{z}\right|^2 dx \\
\leq& 2\int_{\mathbb{R}^N} |\nabla \bar{z}_{t_n} - U|^2 \, dx + 2\int_{\mathbb{R}^N} |\nabla \bar{z} - U|^2 \, dx \\
=& 2\int_{\mathbb{R}^N} \left|t_n^{-1} \nabla \bar{z}(t_n^{-2}x) - U(x)\right|^2 dx + 2\int_{\mathbb{R}^N} |\nabla \bar{z} - U|^2 \, dx \\
\leq& 8t_n^{-1} \int_{\mathbb{R}^N} \left|U(t_n^{-2}x) - U(x)\right|^2 dx + 8\left|t_n^{-1} - 1\right|^2 \int_{\mathbb{R}^N} |U|^2 \, dx \\
&+ \frac{(1 + 2t_n^{2N-2})\varepsilon_0}{10} = \frac{3}{10}\varepsilon_0 + o(1)
\end{aligned} \tag{2.31}$$

and

$$\begin{aligned}
\|\bar{z}_{t_n} - \bar{z}\|_2^2 =& \int_{\mathbb{R}^N} \left|t_n \bar{z}(t_n^{-2}x) - \bar{z}\right|^2 dx \\
\leq& 2\int_{\mathbb{R}^N} |\bar{z}_{t_n} - v|^2 \, dx + 2\int_{\mathbb{R}^N} |\bar{z} - v|^2 \, dx \\
=& 2\int_{\mathbb{R}^N} \left|t_n \bar{z}(t_n^{-2}x) - v(x)\right|^2 dx + 2\int_{\mathbb{R}^N} |\bar{z} - v|^2 \, dx \\
\leq& 8t_n^2 \int_{\mathbb{R}^N} \left|V(t_n^{-2}x) - v(x)\right|^2 dx + 8\left|t_n - 1\right|^2 \int_{\mathbb{R}^N} |v|^2 \, dx \\
&+ \frac{(1 + 2t_n^{2N+2})\varepsilon_0}{10} = \frac{3}{10}\varepsilon_0 + o(1).
\end{aligned} \tag{2.32}$$

From (2.31) and (2.32), we can see

$$\|\bar{z}_{t_n} - \bar{z}\|^2 = \|\nabla \bar{z}_{t_n} - \nabla \bar{z}\|_2^2 + \|\bar{z}_{t_n} - \bar{z}\|_2^2 \leq \frac{3}{5}\varepsilon_0 + o(1),$$

which contradicts with (2.29). So (2.28) holds. Hence, there exists $\delta_1 > 0$ such that

$$|t-1| < \delta_1 \ \Rightarrow \ \|\bar{z}_t - \bar{z}\| < \delta. \tag{2.33}$$

In view of Corollary 2.3, we have for all $t \geq 0$,

$$\begin{aligned} I_\infty(\bar{z}_t) \leq & I_\infty(\bar{z}) - \frac{2-(N+1)t^{2N-2}+(N-1)t^{2N+2}}{2N+2}\|\nabla \bar{z}\|_2^2 \\ = & m_\infty - \frac{2-(N+1)t^{2N-2}+(N-1)t^{2N+2}}{2N+2}\|\nabla \bar{z}\|_2^2 . \end{aligned} \tag{2.34}$$

Let $\varepsilon := \min\left\{\frac{g(0.5)\|\nabla \bar{z}\|_2^2}{4N+4}, \frac{g(1.5)\|\nabla \bar{z}\|_2^2}{4N+4}, 1, \frac{\varrho\delta}{8}\right\}$ and $S := B(\bar{z}, \delta)$, where $g(t)$ is given in (2.11). Then by (Lemma 2.3 in [21]), there exists a deformation $\eta \in \mathcal{C}([0,1] \times E, E)$ such that

(i) $\eta(1,z) = z$ if $I_\infty(z) < m_\infty - 2\varepsilon$ or $I_\infty(z) > m_\infty + 2\varepsilon$;

(ii) $\eta(1, I_\infty^{c+\varepsilon} \cap B(\bar{z},\delta)) \subset I_\infty^{c-\varepsilon}$;

(iii) $I_\infty(\eta(1,z)) \leq I_\infty(z), \ \ \forall\, z \in E$;

(iv) $\eta(1,z)$ is homeomorphism of E.

Form Corollary 2.4, $I_\infty(\bar{z}_t) \leq I_\infty(\bar{z}) = m_\infty$ for $t \geq 0$, then it follows from (2.33) and (ii) that

$$I_\infty(\eta(1,\bar{z}_t)) \leq m_\infty - \varepsilon, \quad \forall\, t \geq 0, \quad |t-1| < \delta_1. \tag{2.35}$$

On the other hand, by (2.11), (2.34) and (iii), we have

$$\begin{aligned} I_\infty(\eta(1,\bar{z}_t)) \leq & I_\infty(\bar{z}_t) \\ \leq & m_\infty - \frac{g(t)}{2N+2}\|\nabla \bar{z}\|_2^2 \\ < & m_\infty, \quad \forall\, t \geq 0, \quad |t-1| \geq \delta_1. \end{aligned} \tag{2.36}$$

Jointly with (2.35) and (2.36), one has

$$\max_{t\in[0.5,1.5]} I_\infty(\eta(1,\bar{z}_t)) < m_\infty. \tag{2.37}$$

We shall prove that $\eta(1,\bar{z}_t) \cap \mathcal{M}_\infty \neq \emptyset$ for some $t \in [0.5, 1.5]$, which contradicts with the definition of m_∞. Let us define

$$\Phi_0(t) := J_\infty(\bar{z}_t), \qquad \Phi_1(t) := J_\infty(\eta(1,\bar{z}_t)), \quad \forall\, t > 0.$$

From Corollary 2.6 and the degree theory, we deduce that $\deg(\Phi_0,\ (0.5,1.5),\ 0) = 1$. By (2.34) and (i), we can see that $\eta(1,\bar{z}_t) = \bar{z}_t$ for $t = 0.5$ and $t = 1.5$. Therefore, $\deg(\Phi_1,\ (0.5,1.5),\ 0) = \deg(\Phi_0,\ (0.5,1.5),\ 0) = 1$. Thus, there exists a $\hat{t} \in (0.5, 1.5)$ such that $\Phi_1(\hat{t}) = 0$, that is $\eta(1,\bar{z}_{\hat{t}}) \in \mathcal{M}_\infty$, which is impossible. $\square$

3 Nonexistence result

In this section, we give some preliminaries and Theorem 1.2 will be proved in the end.

Lemma 3.1. Assume that $(V_1)-(V_3)$ and (G) hold. For any $z \in \mathcal{M}_\infty$, there exists a unique $t > 1$ such that $z_t \in \mathcal{M}$, where $z_t(x) = tz(t^{-2}x)$.

Proof. By (1.12), (1.6) and (2.3), we have for any $\varepsilon > 0$,

$$\begin{aligned} J(z) &> (N-1)\|z\|^2 - \int_{\mathbb{R}^N} (2NG(z) + \nabla G(z))dx \\ &\geq (N-1)\|z\|^2 - \varepsilon \int_{\mathbb{R}^N} |z|^2\, dx - C_\varepsilon \int_{\mathbb{R}^N} |z|^p\, dx \\ &\geq (N-1)\|z\|^2 - \varepsilon\eta_2^2 \|z\|^2 - C_\varepsilon \eta_p^p \|z\|^p. \end{aligned}$$

If we take $\varepsilon = \frac{(N-1)}{2\eta_2^2}$, then there exists $\sigma > 0$ such that

$$J(z) > 0, \quad \forall \;\; 0 < \|z\| \leq \sigma. \tag{3.1}$$

For any $z \in E \setminus \{0\}$, by (2.3), (2.4) and (1.12), we can see

$$J(z) > J_\infty(z), \tag{3.2}$$

which implies $J(z) > 0$ for any $z \in \mathcal{M}_\infty$. Then by (3.1) and Lemma 2.5, there exists a unique $t > 1$ such that $z_t \in \mathcal{M}$. $\square$

Lemma 3.2. Assume that $(V_1) - (V_3)$ and (G) hold. If $z \in \mathcal{M}$, there exists a unique $\tilde{t} \in (0,1)$ such that $z_{\tilde{t}} \in \mathcal{M}_\infty$.

Proof. From (3.2), we can see $J_\infty(z) < 0$ if $z \in \mathcal{M}$. Similar to the proof of (3.1), there exists a $\sigma_1 > 0$ such that $J_\infty(z) > 0$ for $0 < \|z\| \leq \sigma_1$. Therefore, by Corollary 2.6, there exists a unique $\tilde{t} \in (0,1)$ such that $z_{\tilde{t}} \in \mathcal{M}_\infty$. $\square$

Lemma 3.3. Assume that $(V_1)-(V_3)$ and (G) hold. If $z \in \mathcal{M}_\infty$, then $z(\cdot - y) \in \mathcal{M}_\infty$ for any $y \in \mathbb{R}^N$. Moreover, for all $y \in \mathbb{R}^N$, there exists $t_y > 1$ such that $z_{y,t_y}(x) := t_y z(t_y^{-2}(x-y)) \in \mathcal{M}$, and

$$\lim_{|y|\to\infty} t_y = 1.$$

Proof. Let $z = (u,v) \in \mathcal{M}_\infty$, by the translation invariance of I_∞, we can see that $z(\cdot - y) \in \mathcal{M}_\infty$ for any $y \in \mathbb{R}^N$. Moreover, by Lemma 3.1, there exists $t_y > 1$ such that $z_{y,t_y}(x) := t_y z(t_y^{-2}(x-y)) \in \mathcal{M}$, then we have

$$\begin{aligned} &\frac{N-1}{t_y^4}\int_{\mathbb{R}^N} |\nabla z|^2\, dx + (N+1)\int_{\mathbb{R}^N} \left(V(t_y^2 x + y) + \frac{(\nabla V(t_y^2 x + y), t_y^2 x + y)}{N+1} \right) |z|^2\, dx \\ &\quad = \int_{\mathbb{R}^N} \frac{2NG(t_y z) + \nabla G(t_y z)\cdot t_y z}{t_y^2} dx. \end{aligned} \tag{3.3}$$

If $\lim_{|y|\to\infty} t_y = +\infty$, from $(V_1), (V_2), (V_3), (G)$ and Fatou's Lemma, we have

$$(N+1)V_\infty \int_{\mathbb{R}^N} |z|^2 \, dx \geq +\infty, \tag{3.4}$$

which is a contradiction. So we may assume that

$$\lim_{|y|\to\infty} t_y = \alpha \in [1, +\infty) .$$

It follows from (3.1) and $z \in \mathcal{M}_\infty$ that

$$(N-1)\left(\frac{1}{\alpha^4} - 1\right) \|\nabla z\|_2^2 = \int_{\mathbb{R}^N} \left(\frac{2NG(\alpha z) + \nabla G(\alpha z)\cdot \alpha z}{|\alpha z|^2} - \frac{2NG(z) + \nabla G(z)\cdot z}{|z|^2}\right) |z|^2 \, dx,$$

thus, by (G), we can obtain $\alpha = 1$. $\square$

Lemma 3.4. Assume that $(V_1) - (V_3)$ and (G) hold. Then

(i) there exists $\rho > 0$ such that $\|z\| > \rho$ for all $z \in \mathcal{M}$;

(ii) $m := \inf_{z\in\mathcal{M}} I(z) > 0$.

Proof. (i) Let $z \in \mathcal{M}$, that is, $J(z) = 0$, by (1.12), (1.14), (2.3) and Sobolev embedding theorem, we have

$$\begin{aligned}\min\left\{N-1, \frac{(N+1)V_\infty}{\sup_{x\in\mathbb{R}^N} V(x)}\right\} \|z\|^2 &\leq (N-1)\int_{\mathbb{R}^N} |\nabla z|^2 \, dx + (N+1)V_\infty \int_{\mathbb{R}^N} |z|^2 \, dx \\ &\leq \int_{\mathbb{R}^N} (2NG(z) + \nabla G(z)\cdot z) \, dx \\ &\leq \varepsilon \eta_2^2 \|z\|^2 + C_\varepsilon \eta_p^p \|z\|^p .\end{aligned}$$

Set $C_1 := \min\left\{N-1, \frac{(N+1)V_\infty}{\sup_{x\in\mathbb{R}^N} V(x)}\right\}$, if we take $\varepsilon = \frac{C_1}{2\eta_2^2}$, we obtain

$$\|z\| \geq \rho := \left(\frac{C_1}{2C_\varepsilon \eta_p^p}\right)^{\frac{1}{p-2}}, \quad \forall\, z \in \mathcal{M}.$$

(ii) For all $z \in \mathcal{M}$, by (1.12), (1.14) with $\varepsilon \leq (N+1)V_\infty$ and Sobolev embedding inequality, one has

$$\begin{aligned}&(N-1)\int_{\mathbb{R}^N} |\nabla z|^2 \, dx + (N+1)V_\infty \int_{\mathbb{R}^N} |z|^2 \, dx \\ &\leq \varepsilon \int_{\mathbb{R}^N} |z|^2 \, dx + C_\varepsilon \int_{\mathbb{R}^N} |z|^{2^*} \, dx \\ &\leq (N+1)V_\infty \int_{\mathbb{R}^N} |z|^2 \, dx + C' \left(\int_{\mathbb{R}^N} |\nabla z|^2 \, dx\right)^{\frac{N}{N-2}},\end{aligned}$$

which implies that $\int_{\mathbb{R}^N} |\nabla z|^2 \, dx \geq \left(\frac{N-1}{C'}\right)^{\frac{N-2}{2}}$. Hence

$$I(z) = I(z) - \frac{1}{N+1} J(z) \geq \frac{1}{N+1}\int_{\mathbb{R}^N} |\nabla z|^2 \, dx \geq \frac{1}{N+1}\left(\frac{N-1}{C'}\right)^{\frac{N-2}{2}} > 0. \quad \square$$

Lemma 3.5. Assume that $(V_1)-(V_3)$ and (G) hold. Then $m = m_\infty$.

Proof. From Lemma 2.9 and Lemma 2.10, let ω be the ground state solution of system (1.9), that is, $\omega \in \mathcal{M}_\infty$ and $I_\infty(\omega) = m_\infty$. For all $y \in \mathbb{R}^N$, we define $\omega_y(x) := \omega(x-y)$. By the translation invariance of the integrals, we have $\omega_y \in \mathcal{M}_\infty$ and $I_\infty(\omega_y) = m_\infty$. From Lemma 3.3, for any $y \in \mathbb{R}^N$, there exists a $t_y > 1$ such that $\hat{\omega}_y(x) := \omega_{y,t_y}(x) = t_y\omega(t_y^{-2}(x-y)) \in \mathcal{M}$. Therefore, one has

$$\begin{aligned}
|I(\hat{\omega}_y) - m_\infty| =& |I(\hat{\omega}_y) - I_\infty(\omega_y)| \\
=& \left|\frac{1}{2}(t_y^{2N-2}-1)\int_{\mathbb{R}^N} |\nabla\omega_y|^2\,dx + \frac{1}{2}\int_{\mathbb{R}^N}(t_y^{2N+2}V(t_y^2x+y)-V_\infty)\,|\omega_y|^2\,dx \right.\\
&\left. -t_y^{2N}\int_{\mathbb{R}^N} G(t_y\omega_y)dx + \int_{\mathbb{R}^N} G(\omega_y)dx\right| \\
\leq& \frac{1}{2}\left|t_y^{2N-2}-1\right|\int_{\mathbb{R}^N}|\nabla\omega_y|^2\,dx + \frac{1}{2}\left|t_y^{2N+2}V(t_y^2x+y)-V_\infty)\right|\int_{\mathbb{R}^N}|\omega_y|^2\,dx \\
&+\int_{\mathbb{R}^N}\left|G(\omega_y)-t_y^{2N}G(t_y\omega_y)\right|dx.
\end{aligned}$$

Since $t_y \to 1$ and $V(x+y) \to V_\infty$ as $|y| \to \infty$, we obtain

$$I(\hat{\omega}_y) \to m_\infty \text{ as } |y| \to \infty.$$

Hence, we have $m = \inf_{z\in\mathcal{M}} I(z) \leq m_\infty$.

On the other hand, for all $z \in \mathcal{M}$, by Lemma 3.2, there exists a $\tilde{t} \in (0,1)$ such that $u_{\tilde{t}} \in \mathcal{M}_\infty$, then by (V_2), (V_3) and Corollary 2.4, we have

$$\begin{aligned}
m_\infty \leq& I_\infty(z_{\tilde{t}}) \\
=& \frac{1}{2}\tilde{t}^{2N-2}\int_{\mathbb{R}^N}|\nabla z|^2\,dx + \frac{1}{2}\tilde{t}^{2N+2}V_\infty\int_{\mathbb{R}^N}|z|^2\,dx - \tilde{t}^{2N}\int_{\mathbb{R}^N}G(\tilde{t}z)dx \\
\leq& \frac{1}{2}\tilde{t}^{2N-2}\int_{\mathbb{R}^N}|\nabla z|^2\,dx + \frac{1}{2}\tilde{t}^{2N+2}\int_{\mathbb{R}^N}V(\tilde{t}^2x)\,|z|^2\,dx - \tilde{t}^{2N}\int_{\mathbb{R}^N}G(\tilde{t}z)dx \\
=& I(z_{\tilde{t}}) \leq I(z).
\end{aligned}$$

So we obtain $m \geq m_\infty$ and the proof is completed. □

Proof of Theorem 1.2. We suppose, by contradiction, that there exists $\bar{z} \in \mathcal{M}$, which is a critical point of I at level m. From Lemma 3.2, there exists $\bar{t} \in (0,1)$ such that $\bar{z}_{\bar{t}} \in \mathcal{M}_\infty$. Thus by (1.13) and (G), we obtain

$$\begin{aligned}
m =& I(\bar{z}) = I(\bar{z}) - \frac{1}{2N+2}J(\bar{z}) \\
=& \frac{1}{N+1}\int_{\mathbb{R}^N}|\nabla\bar{z}|^2\,dx - \frac{1}{2N+2}\int_{\mathbb{R}^N}(\nabla V(x),x)\,|\bar{z}|^2\,dx + \frac{1}{N+1}\int_{\mathbb{R}^N}(\frac{1}{2}\nabla G(\bar{z})\cdot\bar{z} - G(\bar{z}))dx \\
>& \frac{1}{N+1}\int_{\mathbb{R}^N}|\nabla\bar{z}|^2\,dx + \frac{1}{N+1}\int_{\mathbb{R}^N}(\frac{1}{2}\nabla G(\bar{z})\cdot\bar{z} - G(\bar{z}))dx \\
>& \frac{1}{N+1}\bar{t}^{2N-2}\int_{\mathbb{R}^N}|\nabla\bar{z}|^2\,dx + \frac{1}{N+1}\bar{t}^{2N}\int_{\mathbb{R}^N}(\frac{1}{2}\nabla G(\bar{t}\bar{z})\cdot\bar{t}\bar{z} - G(\bar{t}\bar{z}))dx \\
=& I_\infty(\bar{z}_{\bar{t}}) - \frac{1}{2N+2}J_\infty(\bar{z}_{\bar{t}}) \\
=& I_\infty(\bar{z}_{\bar{t}}) \geq m_\infty,
\end{aligned}$$

which contradicts with the preceding lemma and the proof is completed. □

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