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Abstract

In this work, we solve nonlinear systems of ordinary differential equations coupled to noisy forcing, commonly used for models of neurons such as the Hodgkin-Huxley equation, over a memristor crossbar based computing system. We demonstrate stability and faithfulness of the distributions even under the effects of nonidealities of the memristors and the system itself. We investigate the properties of the dynamical systems under quantization faithfulness, varying the level of precision of the fixed point integer representation and concluding that 24 bits is enough for solution of the Hodgkin-Huxley equations, demonstrating that our solver can operate with both high precision and achieve speedups with low precision approximate computation.

HUXIN: In-Memory Crossbar Core for Integration of Biologically Inspired Stochastic Neuron Models

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***Abstract*—In this work, we solve nonlinear systems of ordinary differential equations coupled to noisy forcing, commonly used for models of neurons such as the Hodgkin-Huxley equation, over a memristor crossbar based computing system. We demonstrate stability and faithfulness of the distributions even under the effects of nonidealities of the memristors and the system itself. We investigate the properties of the dynamical systems under quantization faithfulness, varying the level of precision of the fixed point integer representation and concluding that 24 bits is enough for solution of the Hodgkin-Huxley equations, demonstrating that our solver can operate with both high precision and achieve speedups with low precision approximate computation.**

***Index Terms*—Memristor, Crossbar, Stochasticity, Stochastic Differential Equation, Vector-Matrix Multiplication, Hodgkin-Huxley.**

I. Introduction

MEMRISTOR crossbar systems are next-generation computing devices for accelerating vector matrix multiplication operations. There has also been a recent trend in stochastic computing using the inherent noise in certain regimes of operation in memristors [1]. Systems which harness both the analog and stochastic computing capabilities of memristors promise to reduce computing time for various kinds of stochastic problems.

Approximate computing via resistive random access memory (RRAM) crossbars has recently becoming an area of interest [2], where accuracy of computations is traded for power efficiency and potentially speed. RRAM crossbars offer matrix vector multiplication in $\mathcal{O}(1)$ time, leading to a great speedup over conventional algorithms implemented over von Neumann computing architecture. Stochastic problems are of particular interest, since they generally require less precise computation and therefore are amenable to memristor hardware. Markov Chain Monte Carlo [3] [4] as well as Bayesian neural networks [5] and stochastic error decoders [6] are examples. Memristors have variations in their switching behaviour caused by a natural process, thus the true randomness can be applied to devices such as random number generators (RNG).

The idea of a stochastic computer was first introduced in [7], in which numbers are represented by the probability of bit "1" appearing in a stream of given binary bits. Normal mathematical operations can be performed by single logic gates, which largely reduces the cost of computations. The essence of such computing methodology is a series of bit-stream representation that is stochastic and bit-wise independent. To generate such a bitstream, random number generators (RNGs) are added to the computing system as overheads. Current RNGs are categorized into pseudo-random number generators, such as linear-feedback shift registers (LFSRs), and true-random number generators, using time-variant or multiple threshold structures [8]. One prominent constraint of directly implementing such RNGs in CMOS is the lack of true randomness. Devices such as LFSRs are pseudo-random which means the bitstreams generated have a high possibility of repeated patterns after a certain period [9].

Given the inherent stochastic switching behaviour of memristors, recent works such as [10] [11] [8] introduced memristors into their RNG structures. The above architectures have been benchmarked against the National Institute of Standards and Technology (NIST) group of tests, and exhibit proven ability to generate bitstreams that are uniformly distributed and least likely to overlap [8].

Monte Carlo simulations for stochastic differential equations are one such kind of stochastic problem. Stochastic differential equations model many kinds of physical phenomena in the presence of noise. In order to find the evolution of the probability distribution of the system, many runs of the system [12] have to be found because the equations are intractable by analytic methods. In previous works, we demonstrated a memristor crossbar system for solving various kinds of stochastic differential equations. Here, we focus on the simulation of a Hodgkin-Huxley type neuron models, which directly model the voltage across a section of membrane in a neuron by considering the flow of ions. In the Hodgkin Huxley (HH) model, the capacitance of the membrane as well as the conductance due to gated and passive ion channels are considered [13]. Simplified models replicate the spiking and limit cycle behavior of the HH model but do not directly represent any physical quantities. See Fig 1 for a diagram of the HH model and its equivalent circuit model.

II. Background

Dynamical system models for biological neurons have a long history in both biology and computing. The Hodgkin-Huxley neuron [14], whose dynamical equation is given in eqn. 4. Simplified models exist that capture many of the dynamical behaviors such as the Fitzhugh-Nagumo model, which we will demonstrate our solver is able to reproduce.

Modelling of stochastic effects can be obtained by adding noise terms to certain equations in the system. Injection of

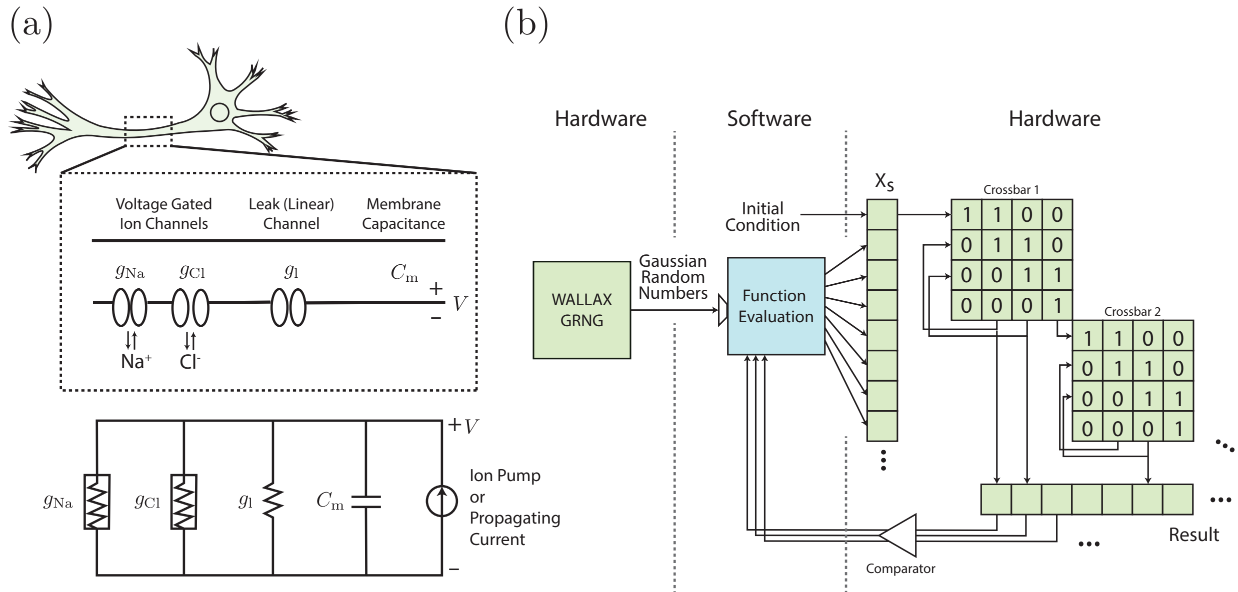

Fig. 1. (a) A Hodgkin-Huxley type model is based off of a phenomenological description of the flow of ions across a section of cell membrane. The different types of transport across the cell membrane can be modelled as circuit components, such as resistors, voltage gated resistors, and current sources. The cell membrane itself is modelled as a capacitor. This leads to a differential equation that models the system. (b) Brief description of hardware design from previous paper. The hardware represents the summation in 1 dimension of the ODE and 1 bit slice. The bit sliced inputs can be input serially and the outputs stored in order to determine the next X_s^i, or multiple crossbars could be for parallel operation.

certain kinds of stochastic behavior can be be obtained by only including stochastic terms in some of the equations, leading to variation in spike times but preservation of the overall shape of the trajectory. The differential equation for the neuron with additive noise can be modelled as a stochastic differential equation, which we will briefly review.

Stochastic differential equations can be considered in two ways: the Ito sense or the Stratonovich sense. Our solver hardware is designed to solve equations formulated in the Ito sense. A general Ito stochastic differential equation is written as

$$dX = \mu(X,t)dt + \sigma(X,t)dW \tag{1}$$

where dW is the derivative of brownian motion. The function μ is the deterministic term and σ is the noise term, which varies the spectral content of the noise. The Euler-Maruyama method is a first order method for solving Ito SDEs, and is written as

$$X(t+\Delta t) - X(t) = \mu(X,t)\Delta t + \sigma(X,t)\Delta W. \tag{2}$$

ΔW has a variance equal to the timestep size dt. This solver converges with accuracy $O(\Delta t)$ in the weak sense (the expected value of the moments converge to the true solution with this order) and $O(\Delta t^{1/2})$ in the strong sense (where the probability density must converge to the true solution).

III. Solution

We use the hardware architecture proposed in Figure 1. The system uses both software and hardware components which divide the computational workload. The software component computes the deterministic functions in the differential equation. The hardware generates the gaussian random numbers and performs the summation for the iterations of the solver.

The gaussian random number generator was described in a previous paper [15], and uses the Wallace method to mix an initial pool of numbers drawn from a gaussian distribution. The matrices that mix the numbers are implemented on crossbars, resulting in fast generation of gaussian random numbers. The software component of the system uses the standard Box-Muller transform to generate the inital pool. The hardware iteratively solves the stochastic differential equation by feeding the current estimate of the solution path into the coefficient matrix and updating the next component of the path successively. The hardware takes advantage of precision extension to increase the accuracy of the calculations. The timeseries is also process sliced into sections, where the integral is sliced into groups eight time steps long. The hardware is modelled using a custom memristor crossbar model [Github link removed for anonymity]. We implement bit slicing and gaussian random number generation on top using the matrix multiplication simulation. The bit slicing technique uses a fixed point integer representation of inputs. We specify the number of integer bits and fraction bits to use, the sum of which is the total number of bits for the representation. This is equivalent to setting the scale of the representation to $2^{\text{fraction bits}}$. Specific problems require different amounts of precision, and we will investigate the effects of the fixed point representation on the FitzHugh-Nagumo dynamics.

The memristors were modelled as resistors with set linear resistance levels between 500Ω and 10kΩ. The line resistance was modelled as 20Ω, and the input and output resistances

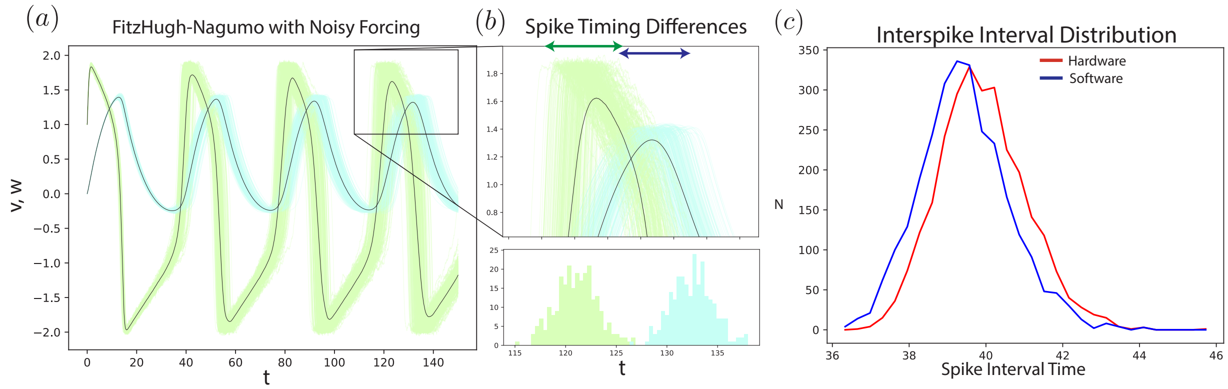

Fig. 2. (a) The dynamics of 250 trajectories of the FitzHugh-Nagumo neuron with noisy forcing. The noise leads to variations in spike timing, which leads to the increased envelope of the dynamical variables. The v variable is shown in green. The w is shown in blue. The means of the trajectories are shown in black. (b) The spike timing differences for the fourth spike are shown up close. Below, the histogram of the times the maximum was achieved for the trajectory in this spike. (c) The interspike interval distribution aggregated for 250 runs using the hardware and software solvers. There is a slight skew in the distribution but the shape is the same, indicated that there is no significant change in the spiking behavior of the neuron at this level of precision.

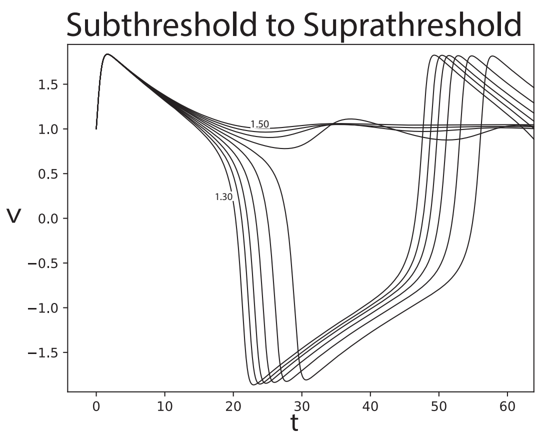

Fig. 3. The bifurcation of the behavior of the neuron due to the parameter b. At 1.3, there is strong spiking behavior. At 1.5, there is a decaying oscillation.

were 1kΩ. The ADCs were modelled with 3 bits in accordance with the operation of the memristors as binary bits in an 8 by 8 crossbar array.

When solving the Hodgkin Huxley system of equations, constraints are imposed on the m, n, and h variables, which represent probabilities. To constrain these variables, we add a digital comparator to our design (Figure 1) which compares the bitstream to the maximum and minimum values allowed to the problem. In our Hodgkin-Huxley model, we impose set boundary conditions where any value of n, m, or h above 1 or below 0 are set to 1 or 0, respectively.

IV. Applications

A. FitzHugh-Nagumo Neuron

Here we demonstrate integration of the FitzHugh-Nagumo model of a neuron, a simplified version of the Hodgkin Huxley neuron. The model is a generalized version of the Van-der-Pol oscillator. This 2-D system is modified with noise [16] [17] in one of the dimensions, leading to differences in spike timing. The equation is as follows:

$$\begin{cases} \dot{v} = v - v^3/3 - w + RI + \text{noise} \\ \dot{w} = v/\tau + a/\tau - bw/\tau \end{cases} \tag{3}$$

For the noise term we use noise $= \sigma dW$ where σ is a constant. To verify the effectiveness of our hardware for simulating this system of stochastic differential equations, we observe the spreading of spike timing differences due to the noise, which increase with the length of the integration. The neuron was run with the parameters $R = 1$, $I = 0.5$, $a = 0.7$, $b = 0.8$, and $T = 12.5$, and the mean dynamics are verified to have spiking behavior. The simulator used 24 fractional bits and 4 integer bits with a process slice of 8 time steps. Each memristor was used as a binary weight.

We compare the interspike interval distribution (ISI) to the same model solved on a regular computer. The plots of the spike timing distribution are shown in Figure 2(c). The computer was used to simulate the Fitzhugh-Nagumo neuron with the same parameters as the hardware. A slight bias is introduced between the hardware and software, with the spike timing being smaller on average than the hardware solver. The shape of the distribution is unaffected, as both are qualitatively unimodal with a leftward skew.

Our solver also allows us to probe the dynamics in the neighborhood of a bifurcation. The FitzHugh-Nagumo neuron undergoes a Hopf bifurcation in the b-parameter [18], the effect on the dynamics we show in Figure 2(d). We turn off the noise and observe the bifurcation of the behavior of the neuron due to the parameter b over the range of 1.3 to 1.5. The neuron does not spike for values above roughly 1.45.

B. Hodgkin-Huxley Neuron

As discussed above, the Hodgkin-Huxley neuron models the flow of ions in a giant squid axon. The equation is an effective

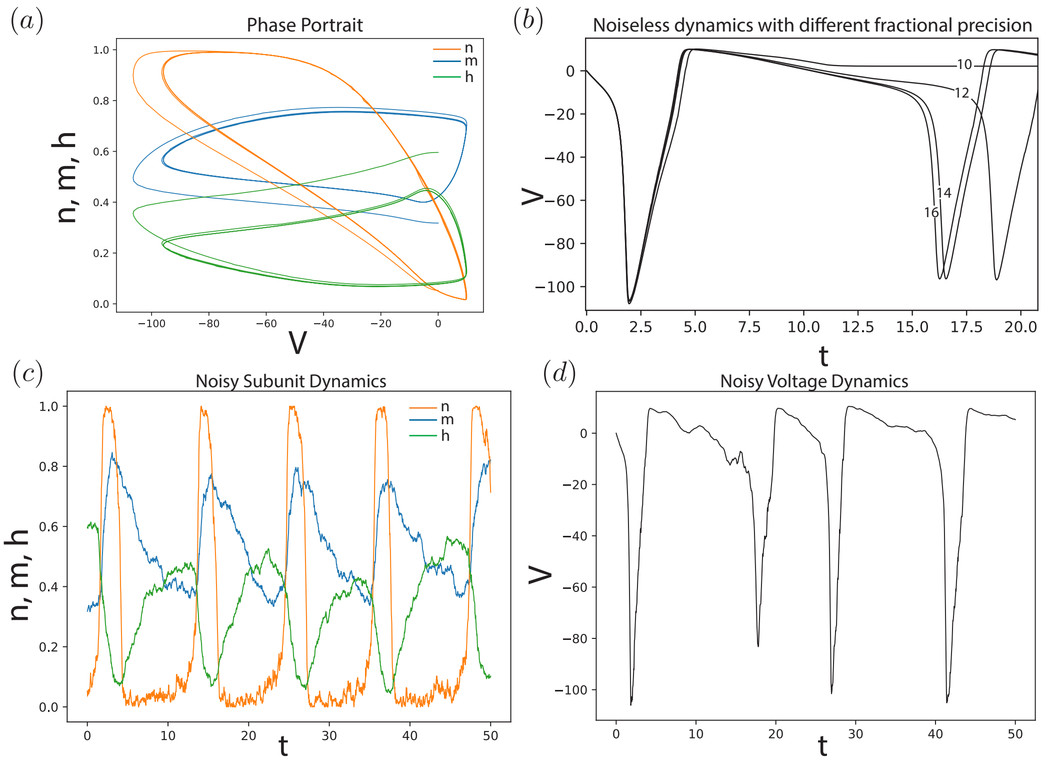

Fig. 4. (a) The phase portrait of the state variables vs. the voltage. (b) The dynamics suffer when the amount of fractional precision used in the solver drops. At 16 bits the dynamics are qualitatively correct, however with 10 bits the spiking behavior is lost. (c) Addition of sub unit noise changes the spike trains from periodic spiking to random spike times. (d) This is pronounced here, where the voltage peaks are not as high and are erratically timed.

model of the ion dynamics, since it does not directly model the ion kinetics. The equation for the voltage is

$$\begin{aligned}\dot{V} = -\frac{1}{C_m}(I(t) &+ g_{\mathrm{K}} m^3 h(V - V_{\mathrm{K}}) \\ &+ g_{\mathrm{Na}} n^4 (V - V_{\mathrm{Na}}) \\ &+ g_l(V + V_l))\end{aligned}$$

with auxiliary state variables

$$\begin{cases} \dot{n} = \alpha_n(V_m)(1-n) - \beta_n(V_m)n + \text{noise} \\ \dot{m} = \alpha_m(V_m)(1-m) - \beta_m(V_m)m + \text{noise} \\ \dot{h} = \alpha_h(V_m)(1-h) - \beta_h(V_m)h + \text{noise} \end{cases} \tag{4}$$

which are constrained by the dynamics (without noise) to be between 0 and 1. The Hodgkin Huxley equations with this type of noise are referred to as having subunit noise [13]. In this case we add white noise, but other noise spectra are possible and lead to different dynamics [19]. In the presence of noise, the derivatives cannot be guaranteed to be negative when the state variable is 1 and positive when the state variable is 0. When the state dynamics pass outside of the bounds the equation becomes unstable. We put clamps on these variables to prevent this. An alternative would be to attempt to take the time step again with a different realization of the noise in until an acceptable step is found.

We investigate the effect of our integer approximation scheme on the dynamics of the Hodgkin Huxley Neuron. Since the voltage variable never goes below -256V, 8 bits are sufficient for the integer portion. We then vary the number of fractional bits and observe the effect of the solved dynamics. The dynamics of the Hodgkin Huxley suffer noticeably below 14 bits. Going from 14 to 16 bits reduces the time to the first spike slightly. Reducing the number of bits allows for a trade-off between speed and accuracy, up to a point.

V. Conclusion

We have demonstrated that our hardware is capable of solving with very high precision nonlinear and possible stochastic neuron models. In solving the Fitzhugh-Nagumo and Hodgkin Huxley models with arbitrary precision, we have demonstrated the utility of in memory resistive RAM computational systems for problems requiring tight tolerances. We also demonstrate how we can reduce the precision by changing our bit slicing for less fractional bits. When neuron models were first introduced, their dynamics were solved via analog computing using discrete circuit components. With the advent of digital computers, densely integrated transistors represented the functions and constants in the differential equations [12]. Here we demonstrate the utility of modern hybrid analog and digital computing for accelerating solvers for nonlinear stochastic differential equations.

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