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A SIMPLE DIRECT PROOF OF THE ERDÖS–STRAUS CONJECTURE

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Abstract. In this note, we prove that the Erdös–Straus conjecture holds true. Similarly, the Sierpinski conjecture follows. A relax extension of the restricted Hagedorn equation is presented.

1. The Erdös–Straus conjecture

The Erdös–Straus conjecture is one of the celebrated open problem in number theory. The conjecture states that, for every positive integer $n \geq 2$, there exist positive integers x, y, z, for which

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \tag{1.1}$$

Similarly, Sierpinski conjectured that the fraction $\frac{5}{n}$ could be written as the sum of three unit fractions. These conjectures are verified for several high order integer. But the question about the proof of these celebrated conjectures are still unsolved open problem, cf. [4].

Hagedorn [5], solved a very special interesting problem concerning $\frac{3}{n}$ under Modular Arithmetic. Namely, Hagedorn proved that for $n > 3$ odd integer not divisible by 3. Then there exist distinct odd, positive integers a, b, and c such that

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}. \tag{1.2}$$

There are a lot of extensive works about this and similar problems, cf. [1], [2], [3], [6], and [7].

In this note, we prove that the Erdös–Straus conjecture holds true. Most famous proofs for all such problems are solved using Modular Arithmetic. However, our presented proof based on very simple tools of analysis. Similar conjectures are also discussed.

2. The Proof of the Erdös–Straus conjecture

Let us observe first that, if $n = 1$ then (1.1) fails. This is obvious since

$$4 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{2.1}$$

does not hold whence $x, y, z \in \mathbb{N}$.

For $n \geq 2$, our proof is carried by contradiction. Suppose there is no integers x, y, z, such that $n(xy + xz + yz) = 4xyz$. Consider the set

$$A := \{n(xy + xz + yz) : \forall x, y, z \in \mathbb{N}, \text{ for some } n \geq 2\}.$$

By our assumption, either $n(xy + xz + yz) < 4xyz$, or $n(xy + xz + yz) > 4xyz$, for all $x, y, z \in \mathbb{N}$ and $n \geq 2$.

Let us assume $n(xy + xz + yz) < 4xyz$. Therefore, the set A is bounded above, and by the least upper bound property the $\sup(A) := \alpha$, exists and in this case, $\alpha = 4xyz$. This implies that

$$\alpha - xy - xz - yz < \alpha,$$

with the properties

(1) $\alpha - xy - xz - yz$ is not an upper bound of A.
(2) $\alpha - xy - xz - yz \in \mathbb{N}$.

This suggests two cases:

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- **Case I:** There exists an integer k such that $\alpha - xy - xz - yz < k < \alpha$, with the property that $k = n\,(xy + xz + yz)$, for some $n \geq 2$. Thus,
$$\alpha - xy - xz - yz < n\,(xy + xz + yz) < \alpha,$$
and this implies that
$$\alpha < n\,(xy + xz + yz) + xy + xz + yz = (n+1)\,(xy + xz + yz) \in A,$$
which contradicts the assumption that α is the least upper bound of A.

- **Case II:** There exists an integer k such that $\alpha - xy - xz - yz < k < \alpha$, with the property that $k \neq n\,(xy + xz + yz)$, for some $n \geq 2$. But thus, there exists $m \in \mathbb{N}$ with $m \geq 2$ with the property that $\alpha - xy - xz - yz < k < m\,(xy + xz + yz)$. Hence, we reach a contradiction as the previous step.
  Similar contradictions occurred, if $n\,(xy + xz + yz) > 4xyz$. So that, there exist $x, y, z \in \mathbb{N}$ such that $n\,(xy + xz + yz) = 4xyz$, for some $n \geq 2$, which completes the proof. Thus, the Erdös–Straus conjecture $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, holds true for some integers x, y, z and $n \geq 2$.

In the previous presented proof, nothing special about $4/n$. Replacing, $4/n$ by $5/n$, then the Sierpinski conjecture holds true using the same proposed argument. Furthermore, the restricted Hagedorn equation (1.2) requires that n odd and it is not divisible by 3. For example, the following two cases are not guaranteed by Hagedorn assumptions. Take $n = 8$ and $a = b = c = \frac{1}{8}$, we note that
$$\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}, \quad \text{(or)} \quad \frac{3}{8} = \frac{1}{16} + \frac{1}{4} + \frac{1}{16}.$$
Also, take $n = 9$ and $a = b = c = \frac{1}{9}$, we note that
$$\frac{3}{9} = \frac{1}{3} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9}, \quad \text{(or)} \quad \frac{1}{3} = \frac{1}{18} + \frac{1}{9} + \frac{1}{6}.$$
In fact, these cases (when n is not odd and/or divisible by 3) together with the other assumptions treated by Hagedorn himself could be very obvious if one follows the same approach of proving the Erdös–Straus conjecture above. Indeed, our approach could be very useful for refining and relaxing the original problem of Hagedorn [5], and in the meanwhile covers the remaining cases all together without any further restrictions as proposed in [5].

Roughly, and in general, for a fixed positive integer m, the Diophantine equation $n\,(xy + xz + yz) = mxyz$, or we write $\frac{m}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, has a solution for some positive integers x, y, z and $n \geq 2$, as long as m is a fixed positive integer.

Finally, we reformulate (refine) and restrict the original conjecture of Erdös–Straus, to be as in the following form.

Problem 1. *Prove the Erdös–Straus conjecture using Modular Arithmetic only.*

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