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Abstract

In this note, we prove that the Erdös-Straus conjecture holds true. Similarly, the Sierpinski conjecture follows. A relaxed extension of the restricted Hagedorn equation is presented.

Main File

A SIMPLE DIRECT PROOF OF THE ERDÖS–STRAUS CONJECTURE

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Abstract. In this note, we give full complete positive proof of the celebrated unsolved Erdös–Straus conjecture. Similarly, the Sierpinski conjecture follows. A relaxed extension of the restricted Hagedorn equation is presented.

1. The Erdös–Straus conjecture

The Erdös–Straus conjecture is one of the celebrated unsolved open problem in number theory. The conjecture states that, for every positive integer $n \geq 2$, there exist positive integers x, y, z, for which

$$\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}. \tag{1.1}$$

Similarly, Sierpinski conjectured that the fraction $\frac{5}{n}$ could be written as the sum of three unit fractions. These conjectures are verified for several high order integer. For example, Swett [7] has established the validity of (1.1) for all $n \leq 10^{14}$ and Salez [8] for all $n \leq 10^{17}$. But the question about the proof of these celebrated conjectures are still unsolved open problem, cf. [4].

There are many attempts to solve this conjecture. As of writing this note, no complete proof of this conjecture has been provided and the problem is still an unsolved open problem. We have even read some glossy headlines claiming to have solved this conjecture, but in fact, no one has yet. Some researchers claimed to refute the Erdös–Straus conjecture but this is also, not true.

Most of the published attempts are based on Modular Arithmetic and all such attempts offer a partial solution for some exceptional cases of the original conjecture.

No one presented any full completed proof of the original conjecture as it was characterized by Erdös–Straus; *which was argue the existence of integer solution of* (1.1) *for* $n \geq 2$. This is the original problem nothing else.

Hagedorn [5], solved a very special interesting problem concerning $\frac{3}{n}$ under Modular Arithmetic. Namely, Hagedorn proved that for $n > 3$ odd integer not divisible by 3. Then there exist distinct odd, positive integers a, b, and c such that

$$\frac{3}{n} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}. \tag{1.2}$$

There are a lot of extensive works about this and similar problems, cf. [1], [2], [3], [6], and [9].

In this note, we give a full complete positive proof of the celebrated unsolved Erdös–Straus conjecture. Our presented proof based on very simple tools of analysis. Similar conjectures are also discussed.

2. The Proof of the Erdös–Straus conjecture

Let us observe first that, if $n = 1$ then (1.1) fails. This is obvious since

$$4 = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \tag{2.1}$$

does not hold whence $x, y, z \in \mathbb{N}$.

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Proof. For $n \geq 2$, our proof is carried by contradiction. Suppose there is no integers x, y, z, such that $n(xy + xz + yz) = 4xyz$. Consider the set

$$A := \{n(xy + xz + yz) : \forall x, y, z \in \mathbb{N}, \text{ for some } n \geq 2\}.$$

By our assumption, either $n(xy + xz + yz) < 4xyz$, or $n(xy + xz + yz) > 4xyz$, for all $x, y, z \in \mathbb{N}$ and $n \geq 2$.

Let us assume $n(xy + xz + yz) < 4xyz$. Therefore, the set A is bounded above, and by the least upper bound property the $\sup(A) := \alpha$, exists and in this case, $\alpha = 4xyz$. This implies that

$$\alpha - xy - xz - yz < \alpha,$$

with the properties

(1) $\alpha - xy - xz - yz$ is not an upper bound of A.
(2) $\alpha - xy - xz - yz \in \mathbb{N}$.

This suggests that there exists an integer k such that $\alpha - xy - xz - yz < k < \alpha$, with the property that $k = n(xy + xz + yz)$, for some $n \geq 2$. Thus,

$$\alpha - xy - xz - yz < n(xy + xz + yz) < \alpha,$$

and this implies that

$$\alpha < n(xy + xz + yz) + xy + xz + yz = (n+1)(xy + xz + yz) \in A,$$

which contradicts the assumption that α is the least upper bound of A. Our proof is finished once we prove such $k \in A$ exists, i.e; there exists an integer k has the form $n(xy + xz + yz)$, for some integers x, y, z and $n \geq 2$ with the property that $\alpha - xy - xz - yz < k$.

To obtain that, assume $x, y, z \in \mathbb{N}$ and without loss of generality assume $x < y < z$. Setting $x = m$, $y = m + p$, $z = m + q$; such that $p, q \in \mathbb{N}$.

$$\begin{aligned}
\alpha - xy - xz - yz &= 4xyz - xy - xz - yz \\
&= 4m^3 + 4m^2q + 4pm^2 + 4mpq - 3m^2 - 2pm - 2mq - pq \\
&= 4m^3 + 4m(mq + pm + pq) - 2m^2 - 2pm - 2mq - m^2 - pq \\
&< 4m^3 \cdot (mq + pm + pq) + 4m(mq + pm + pq) \\
&= \left(4m^3 + 4m\right) \cdot (mq + pm + pq) \\
&= N \cdot (mq + pm + pq) \in A,
\end{aligned}$$

which proves such k exists.

A similar contradiction occurred, when we assume $n(xy + xz + yz) > 4xyz$. So that, there exist $x, y, z \in \mathbb{N}$ such that $n(xy + xz + yz) = 4xyz$, for some $n \geq 2$, which completes the proof. Thus, the Erdös–Straus conjecture $\frac{4}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, holds true for some integers x, y, z and $n \geq 2$. □

In the previous presented proof, nothing special about $4/n$. Replacing, $4/n$ by $5/n$, then the Sierpinski conjecture holds true using the same proposed argument. Furthermore, the restricted Hagedorn equation (1.2) requires that n odd and it is not divisible by 3. For example, the following two cases are not guaranteed by Hagedorn assumptions. Take $n = 8$ and $a = b = c = \frac{1}{8}$, we note that

$$\frac{3}{8} = \frac{1}{8} + \frac{1}{8} + \frac{1}{8}, \quad \text{(or)} \quad \frac{3}{8} = \frac{1}{16} + \frac{1}{4} + \frac{1}{16}.$$

Also, take $n = 9$ and $a = b = c = \frac{1}{9}$, we note that

$$\frac{3}{9} = \frac{1}{3} = \frac{1}{9} + \frac{1}{9} + \frac{1}{9}, \quad \text{(or)} \quad \frac{1}{3} = \frac{1}{18} + \frac{1}{9} + \frac{1}{6}.$$

In fact, these cases (when n is not odd and/or divisible by 3) together with the other assumptions treated by Hagedorn himself could be very obvious if one follows the same approach of proving the Erdös–Straus conjecture above. Indeed, our approach could be very useful for refining and relaxing the original problem of Hagedorn [5], and in the meanwhile covers the remaining cases all together without any further restrictions as proposed in [5].

Roughly, and in general, for a fixed positive integer m, the Diophantine equation $n(xy + xz + yz) = mxyz$, or we write $\frac{m}{n} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$, has a solution for some positive integers x, y, z and $n \geq 2$, as long as m is a fixed positive integer.

Finally, we reformulate (refine) and restrict the original conjecture of Erdös–Straus, to be as in the following form.

Problem 1. *Prove the Erdös–Straus conjecture using Modular Arithmetic only.*

Data Availability. The data used to support the findings of this study are included within the article.

Conflicts of Interest. The author declares that he has no conflicts of interest.

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