Well-posedness of QSDEs driven by fermion Brownian motion in noncommutative Lp -space

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This paper is concerned with quantum stochastic differential equations driven by the fermion field in noncommutative space $L^p(\mathscr{C})$ for $p > 2$. We investigate the existence and uniqueness of L^p-solution of quantum stochastic differential equations in infinite time horizon by the Burkholder-Gundy inequalities for noncommutative martingales given by Pisier and Xu. Finally, we obtain Markov property and the self-adjointness which is of great significance in the study of optimal control problems.
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Keywords: Quantum stochastic differential equations; Fermion Brownian motion; The Burkholder-Gundy inequality; Noncommutative L^p-space.

1 Introduction

In the present paper, we shall consider the existence, uniqueness and other related properties of solution of the following quantum stochastic differential equation (QSDE for short) driven by the fermion field introduced by Barnett, Streater and Wilde [14] in noncommutative space $L^p(\mathscr{C})$:

$$dX_t = F(X_t, t)dW_t + dW_t G(X_t, t) + H(X_t, t)dt, \ t \geq 0, \tag{1.1}$$

which is closely related to the quantum noise, quantum fields etc. Quantum stochastic integration and quantum stochastic differentiation are actively used in quantum optics, quantum measurement theory and quantum filtering theory [10, 11]. Therefore, it is significant in theory and application to investigate QSDEs. The fermion field and the boson field are the two most important quantum fields. These QSDEs driven by fermion fields and boson fields respectively, can be uniformly understood as same framework of the Hudson and Parthasarathy's quantum stochastic calculus [19, 24] in noncommutative spaces. There are many efforts to study solutions

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of QSDEs by many mathematicians [1, 4, 8, 9, 20, 23, 27, 30–32] and reference therein. In [22], Hudson and Parthasarathy studied the unitary solution of a special class of QSDE

$$dU = U\left(\sum_j \{L_j dA - L_j dA^*\} + \left(i\mathbb{H} - \frac{1}{2}\sum_j L_j L_j^*\right) dt\right),\ U(0) = I. \tag{1.2}$$

The weak solutions of the following QSDE

$$dM_t = F(M_t, t)dA_t + dA_t^* G(M_t, t) + H(M_t, t)dt,\ t_0 \leq t \leq T, \tag{1.3}$$

with initial value condition $M_{t_0} = Z$ was studied in [8, 25, 26], where M_t is a weak solution if

$$M_t u = \left(M_{t_0} + \int_{t_0}^t F(M_s, s)dA_s + \int_{t_0}^t dA_s^* G(M_s, s) + \int_{t_0}^t H(M_s, s)ds\right) u,\ t_0 \leq t \leq T,$$

for any $u \in D(M_t)$, where M_t is an unbounded operator with the domain $D(M_t)$. In [29], QSDE (1.3) is called the Heisenberg evolution of the Schödinger equation in the fermion field. By using the isometry property of the Itô-Clifford stochastic integral in noncommutative space $L^2(\mathscr{C})$, Barnett, Streater and Wilde [14] considered the solution of QSDE (1.6) in finite time horizon. Bishop, Okeke and Eka [12] discussed the existence and uniqueness of mild solution of the quantum evolution equation. The aforementioned results are restricted to finite time horizon cases. In this paper, we investigate the L^p-solution of QSDEs in infinite time horizon for $p > 2$.

The noncommutative L^p-spaces and associated Harmonic analysis have been deeply studied in [17, 21, 34–37] and references therein. Let $\mathscr{H}$ be a separable complex Hilbert space. The anti-symmetric Fock space over $\mathscr{H}$ is defined by

$$\Lambda(\mathscr{H}) = \bigoplus_{n=0}^{\infty} \Lambda_n(\mathscr{H}),$$

where $\Lambda_0(\mathscr{H}) = \mathbb{C}$ and $\Lambda_n(\mathscr{H})$ is the Hilbert space anti-symmetric n-fold tensor product of $\mathscr{H}$ with itself. For any $z \in \mathscr{H}$, the creation operator $C(z) : \Lambda_n(\mathscr{H}) \to \Lambda_{n+1}(\mathscr{H})$ defined by $u \mapsto \sqrt{n+1}\ z\wedge u$, is a bounded operator on $\Lambda(\mathscr{H})$ with $\|C(z)\| = \|z\|$. The annihilation operator $A(z)$ is the adjoint of $C(z)$, i.e. $A(z) = C(z)^*$. The fermion field $\Psi(z)$ is defined on $\Lambda(\mathscr{H})$ by

$$\Psi(z) := C(z) + A(Jz),$$

where $J : \mathscr{H} \to \mathscr{H}$ is a conjugation operator (i.e., J is antilinear, antiunitary and $J^2 = 1$). Denote by $\mathscr{C}$ the von Neumann algebra generated by the bounded operators $\{\Psi(z) : z \in \mathscr{H}\}$. For the Fock vacauum $\mathbb{1} \in \Lambda(\mathscr{H})$, define

$$m(\cdot) := \langle \mathbb{1}, \cdot\mathbb{1}\rangle \tag{1.4}$$

on $\mathscr{C}$. Obviously, m is a normal faithful state on $\mathscr{C}$, and $(\Lambda(\mathscr{H}), \mathscr{C}, m)$ is a quantum (noncommutative) probability space by [38]. For any $1 \leq p < \infty$, define the noncommutative L^p-norm on $\mathscr{C}$ by

$$\|f\|_p := m(|f|^p)^{\frac{1}{p}} = \langle \mathbb{1}, |f|^p\mathbb{1}\rangle^{\frac{1}{p}},$$

where $|f| = (f^*f)^{\frac{1}{2}}$, then $L^p(\mathscr{C}, m)$ is the completion of $(\mathscr{C}, \|\cdot\|_p)$, which is the noncommutative L^p-space, abbreviated as $L^p(\mathscr{C})$.

Now, let $\mathscr{H} = L^2(\mathbb{R}^+)$, and J be the complex conjugation on $L^2(\mathbb{R}^+, ds)$. For given $0 \leq t < \infty$, define $\mathscr{C}_t$ to be the von Neumann subalgebra of $\mathscr{C}$ generated by

$$\left\{\Psi(u) : u \in L^2(\mathbb{R}_+),\ \text{ess supp } u \subseteq [0,t]\right\},$$

then $\{\mathscr{C}_t\}_{t\geq 0}$ is an increasing family of von Neumann subalgebras of $\mathscr{C}$ which is the noncommutative analogue of filtration in stochastic analysis. Let

$$W_t := W(\chi_{[0,t]}) = \Psi(\chi_{[0,t]}) = C(\chi_{[0,t]}) + A(J\chi_{[0,t]}),\ t \in \mathbb{R}^+, \tag{1.5}$$

then $\{W_t : t \in \mathbb{R}^+\}$ is the fermion analogue of Brownian motion adapted to the family $\{\mathscr{C}_t : t \in \mathbb{R}^+\}$, which is called fermion Brownian motion.

In the rest of this paper, we mainly consider the following QSDE in infinite time horizon,

$$\begin{cases} dX_t = F(X_t,t)dW_t + dW_tG(X_t,t) + H(X_t,t)dt, \text{ in } [t_0, \infty), \\ X_{t_0} = Z, \end{cases} \tag{1.6}$$

where $F(\cdot,\cdot),\ G(\cdot,\cdot),\ H(\cdot,\cdot) : L^p(\mathscr{C}) \times \mathbb{R}^+ \to L^p(\mathscr{C})$ are operator-valued functions and $Z \in L^p(\mathscr{C}_{t_0})$ for fixed $p > 2$. Since W_t is a bounded operator and $F(x(t),t),\ G(x(t),t) \in L^p(\mathscr{C})$ for any $t \in [t_0, \infty)$, they are not commuting.

Definition 1.1. A stochastic process $X_{(\cdot)} : [t_0,\infty) \to L^p(\mathscr{C})$ is called a solution of QSDE (1.6) if it satisfies

$$X_t = Z + \int_{t_0}^t F(X_s,s)dW_s + \int_{t_0}^t dW_sG(X_s,s) + \int_{t_0}^t H(X_s,s)ds,\ \text{a.s. } t \geq t_0.$$

Throughout the paper, we shall make the following assumptions.

Assumption 1.1. $F(\cdot,\cdot),\ G(\cdot,\cdot),\ H(\cdot,\cdot)$: $L^p(\mathscr{C}) \times \mathbb{R}^+ \to L^p(\mathscr{C})$ are operator-valued functions such that

(A1) $F(\cdot,\cdot),\ G(\cdot,\cdot),\ H(\cdot,\cdot) : L^p(\mathscr{C}) \times \mathbb{R}^+ \to L^p(\mathscr{C})$ are adapted.

(A2) For any $x \in L^p(\mathscr{C})$, $F(x,\cdot),\ G(x,\cdot),\ H(x,\cdot) : \mathbb{R}^+ \to L^p(\mathscr{C})$ are continuous a.s..

(A3) Lipschitz condition: For any $x_1,\ x_2 \in L^p(\mathscr{C})$ and a.e. $t \in [0,\infty)$, there exists a constant K such that

$$\|F(x_1,t) - F(x_2,t)\|_p + \|G(x_1,t) - G(x_2,t)\|_p + \|H(x_1,t) - H(x_2,t)\|_p \leq K\|x_1 - x_2\|_p.$$

(A3') Osgood condition: For any $x_1,\ x_2 \in L^p(\mathscr{C})$ and a.e. $t \in [0,\infty)$,

$$\|F(x_1,t) - F(x_2,t)\|_p^2 + \|G(x_1,t) - G(x_2,t)\|_p^2 + \|H(x_1,t) - H(x_2,t)\|_p^2 \leq \rho(\|x_1 - x_2\|_p^2),$$

where $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous non-decreasing function with $\rho(0) = 0$, $\rho(r) > 0$ for $r > 0$, such that $\int_{0+} \frac{dr}{\rho(r)} = +\infty$.

(A4) For any $t \in [0, \infty)$,

$$F(0,t) = G(0,t) = H(0,t) = 0, \ a.e..$$

Theorem 1.1. *Under the above assumptions, for $p > 2$, there is a unique continuous adapted L^p-solution $\{X_t\}_{t \geq t_0}$ of the following quantum stochastic integral equation,*

$$X_t = Z + \int_{t_0}^{t} F(X_s, s)dW_s + \int_{t_0}^{t} dW_s G(X_s, s) + \int_{t_0}^{t} H(X_s, s)ds, \ t \geq t_0. \tag{1.7}$$

Remark 1.1. The Burkholder-Gundy inequalities for noncommutative martingales of Pisier and Xu have been used by Dirksen [18] to study the L^p-solution of QSDE with respect to any normal L^p-martingale for $p > 2$ without the drift term $\int_{t_0}^{t} H(X_s, s)ds$. Under this condition, the solution of the QSDE is a martingale, so the Burkholder-Gundy inequalities can be used to study the existence and uniqueness of L^p-solution. However, if the drift term exists, then the solution of the QSDE (1.6) is not a martingale. Therefore, the Burkholder-Gundy inequalities are not used to study the solution of QSDE (1.6) directly. In order to prove Theorem 1.1, we establish the following inequality

$$\left\| \int_{0}^{t} f(s)dW_s \right\|_p \leq C(p) \left(\int_{0}^{t} \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}, \ 0 \leq t \leq T, \tag{1.8}$$

based on the Burkholder-Gundy inequalities and Minkowski-type inequality.

This paper is organized as follows. In Section 2, we recall some preliminaries on fermion fields. Sections 3 and 4 are devoted to proving the existence and uniqueness of L^p-solutions of QSDEs in finite time horizon and in infinite time horizon by classical Picard iteration and Banach's fixed-point theorem, respectively. In Section 5, we shall gain the self-adjointness and the Markov property of the solution of QSDEs.

2 Preliminaries and the Burkholder-Gundy inequalities

In this section, we introduce the main techniques to solve problems later. We first recall some notations and concepts [13–15, 33, 39, 40] necessary to the whole paper.

Definition 2.1. A map $X : \mathbb{R}^+ \to L^p(\mathscr{C})$ is said to be adapted if $X_t \in L^p(\mathscr{C}_t)$ for each $t \in \mathbb{R}^+$. A map $F : L^p(\mathscr{C}) \times \mathbb{R}^+ \to L^p(\mathscr{C})$ is said to be adapted if $F(u,t) \in L^p(\mathscr{C}_t)$, for any $t \in \mathbb{R}^+$ and $u \in L^p(\mathscr{C}_t)$.

It is easy to verify that if $X : \mathbb{R}^+ \to L^p(\mathscr{C})$ and $F : L^p(\mathscr{C}) \times \mathbb{R}^+ \to L^p(\mathscr{C})$ are both adapted, so is the map $t \mapsto F(X_t, t)$.

Definition 2.2. An adapted L^p-processes f on $[t_0, t]$ is said to be simple if it can be expressed as

$$f = \sum_{k=0}^{n-1} f(t_k)\chi_{[t_k, t_{k+1})} \tag{2.1}$$

on $[t_0, t]$ for $t_0 \leq t_1 \leq \cdots \leq t_n = t$ and $f(t_k) \in L^p(\mathscr{C}_{t_k})$ for all $0 \leq k \leq n-1$.

By [13], the Itô-Clifford stochastic integral of any simple adapted L^p-process with respect to fermion Brownian motion W_t is defined as follow.

Definition 2.3. If $f = \sum_k f(t_k)\chi_{[t_k,t_{k+1})}$ is a simple adapted L^p-processes on $[t_0, t]$, the Itô-Clifford stochastic integral of f over $[t_0, t]$ with respect to W_t is

$$\int_{t_0}^{t} f(s)dW_s = \sum_{k=0}^{n-1} f(t_k)(W_{t_{k+1}} - W_{t_k}). \tag{2.2}$$

For any $p \geq 2$, let $\mathcal{S}_{\mathbb{A}}^p(\mathbb{R}^+)$ be the linear space of all simple adapted L^p-processes, i.e.

$$\mathcal{S}_{\mathbb{A}}^p(\mathbb{R}^+) := \left\{ f : \mathbb{R}^+ \to L^p(\mathscr{C}),\ f \text{ is simple and adapted} \right\}.$$

Then, $\mathcal{S}_{\mathbb{A}}^p([0,t])$ is subspace of $\mathcal{S}_{\mathbb{A}}^p(\mathbb{R}^+)$ whose processes vanish in (t, ∞). It is clear that $\int_{t_0}^t f(s)dW_s$ is a Clifford L^p-martingale for any $f \in \mathcal{S}_{\mathbb{A}}^p([0,t])$, i.e.

$$\mathbb{E}\left(\int_{t_0}^{t} f(\tau)dW_\tau \middle| \mathscr{C}_s \right) = \int_{t_0}^{s} f(\tau)dW_\tau,\ t_0 \leq s \leq t.$$

For any $f \in \mathcal{S}_{\mathbb{A}}^p([0,t])$, let

$$\|f\|_{\mathcal{H}^p([0,t])} := \max\left\{ \left\| \left(\int_0^t |f(s)|^2 ds \right)^{\frac{1}{2}} \right\|_p,\quad \left\| \left(\int_0^t |f(s)^*|^2 ds \right)^{\frac{1}{2}} \right\|_p \right\},$$

and the noncommutative Hardy space $\mathcal{H}^p([0,t])$ be the completion of $\mathcal{S}_{\mathbb{A}}^p([0,t])$ with the norm $\|\cdot\|_{\mathcal{H}^p([0,t])}$. For simplicity, let $\mathcal{H}_{loc}^p(\mathbb{R}^+)$ be the space of all stochastic processes $f : \mathbb{R}^+ \to L^p(\mathscr{C})$ and $\chi_{[0,t]}f \in \mathcal{H}^p([0,t])$. Moreover, we have the Burkholder-Gundy inequality (2.3) for the Clifford L^p-martingale first established in [33].

Lemma 2.1. *[33, Theorem 4.1] Let $2 \leq p < \infty$. Then, for any $f \in \mathcal{H}_{loc}^p(\mathbb{R}^+)$ and its Itô-Clifford stochastic integral*

$$X_t = \int_0^t f(s)dW_s,\ t \geq 0,$$

it holds that

$$\alpha_p^{-1}\|f\|_{\mathcal{H}^p([0,t])} \leq \|X_t\|_p \leq \beta_p\|f\|_{\mathcal{H}^p([0,t])},\ t \geq 0, \tag{2.3}$$

where α_p and β_p are positive constants depend on p.

The stochastic integral (2.2) is also called right stochastic integral. Analogously, we can define left stochastic integrals $\int_0^t dW_s f(s)$, and have the Burkholder-Gundy inequalities with respect to left stochastic integrals.

Lemma 2.2. *[18, Theorem 7.2] Let $1 < p < \infty$. For any $f \in \mathcal{H}_{loc}^p(\mathbb{R}^+)$, the left stochastic integral $\int_0^t dW_s f(s)$ and right stochastic integral $\int_0^t f(s)dW_s$ are continuous L^p-martingales and the following holds*

$$\left\| \int_0^t dW_s f(s) \right\|_p \simeq_p \|f\|_{\mathcal{H}^p([0,t])} \simeq_p \left\| \int_0^t f(s)dW_s \right\|_p.$$

By Lemma 2.1 and Lemma 2.2, the Itô-Clifford stochastic integral can be defined for any element of $\mathcal{H}^p([0,t])$, and the Burkholder-Gundy inequality (2.3) holds ture. Next, we define several classes Banach spaces of adapted processes in noncommutative space $L^p(\mathscr{C})$. For any interval $[0,T]\subset\mathbb{R}^+$,

$$C_{\mathbb{A}}([0,T];L^p(\mathscr{C})) := \Big\{x(\cdot):[0,T]\to L^p(\mathscr{C}) \mid x(\cdot) \text{ is adapted process} \\ \text{and } \lim_{s\to t}\|x(s)-x(t)\|_p=0,\ 0\le s,t\le T]\Big\}.$$

It is easy to check that $C_{\mathbb{A}}([0,T];L^p(\mathscr{C}))$ is a Banach space equipped with the norm

$$\|x\|_{C_{\mathbb{A}}([0,T];L^p(\mathscr{C}))}=\max_{t\in[0,T]}\|x(t)\|_p.$$

For any $1<p,q<\infty$, let $L^q_{\mathbb{A}}(0,t;L^p(\mathscr{C}))$ be the completion of $\mathcal{S}^p_{\mathbb{A}}(\mathbb{R}^+)$ with the norm

$$\|f\|_{L^q_{\mathbb{A}}(0,t;L^p(\mathscr{C}))}=\left(\int_0^t\|f(s)\|_p^q ds\right)^{\frac{1}{q}},\quad t\ge 0.$$

Similarly, $L^p_{\mathbb{A}}(\mathscr{C};L^q(0,t))$ is the completion of $\mathcal{S}^p_{\mathbb{A}}([0,t])$ with the norm

$$\|f\|_{L^p_{\mathbb{A}}(\mathscr{C};L^q(0,t))}=\left\|\left(\int_0^t|f(s)|^q ds\right)^{\frac{1}{q}}\right\|_p,\quad t\ge 0.$$

As an application of Minkowski-type inequality, we give important inequalities on the above Banach spaces.

Theorem 2.3. *Let $1<q\le p<\infty$. Then, for any $f\in L^q_{\mathbb{A}}(0,T;L^p(\mathscr{C}))$,*

$$\left\|\left(\int_0^t|f(s)|^q ds\right)^{\frac{1}{q}}\right\|_p\le\left(\int_0^t\|f(s)\|_p^q ds\right)^{\frac{1}{q}},\quad 0\le t\le T. \tag{2.4}$$

Furthermore, $L^q_{\mathbb{A}}(0,T;L^p(\mathscr{C}))\subseteq L^p_{\mathbb{A}}(\mathscr{C};L^q(0,T))$.

Proof. Let

$$0=t_0\le t_1\le t_2\le\cdots\le t_n=t$$

be an equal time partition of $[0,t]$ where the mesh of the subdivision is $l=t/n=t_{k+1}-t_k$, $k=0,\ 1,\ \cdots,\ n-1$. For simple adapted process $\sum_{k\ge0}a_{t_k}\chi_{[t_k,t_{k+1})}$ of $L^p(\mathscr{C})$ where $a_{t_k}\in L^p(\mathscr{C}_{t_k})$, and any positive integer n, one has

$$\left\|\left(\sum_{k=0}^{n-1}|a_{t_k}|^q(t_{k+1}-t_k)\right)^{\frac{1}{q}}\right\|_p=l^{\frac{1}{q}}\left\|\left(\sum_{k=0}^{n-1}|a_{t_k}|^q\right)^{\frac{1}{q}}\right\|_p=l^{\frac{1}{q}}\left\|\sum_{k=0}^{n-1}|a_{t_k}|^q\right\|_{\frac{p}{q}}^{\frac{1}{q}}. \tag{2.5}$$

Since $\frac{p}{q}\ge1$, by Minkowski inequality [41, Theorem 5.2.2],

$$\left\|\sum_{k=0}^{n-1}|a_{t_k}|^q\right\|_{\frac{p}{q}}\le\sum_{k=0}^{n-1}\||a_{t_k}|^q\|_{\frac{p}{q}},\quad n\in\mathbb{N}^+. \tag{2.6}$$

From (2.5) and (2.6), we have

$$\left\| \left(\sum_{k=0}^{n-1} |a_{t_k}|^q (t_{k+1} - t_k) \right)^{\frac{1}{q}} \right\|_p \leq \left(\sum_{k=0}^{n-1} \|a_{t_k}\|_p^q (t_{k+1} - t_k) \right)^{\frac{1}{q}}.$$

Since $\mathcal{S}_{\mathbb{A}}^p(\mathbb{R}^+)$ is dense in $L_{\mathbb{A}}^p(\mathscr{C}; L^q(0,T))$,

$$\left\| \left(\int_0^t |f(s)|^q ds \right)^{\frac{1}{q}} \right\|_p \leq \left(\int_0^t \|f(s)\|_p^q ds \right)^{\frac{1}{q}}, \quad 0 \leq t \leq T.$$

for any $f \in L_{\mathbb{A}}^q(0,T; L^p(\mathscr{C}))$, and $L_{\mathbb{A}}^q(0,T; L^p(\mathscr{C})) \subseteq L_{\mathbb{A}}^p(\mathscr{C}; L^q(0,T))$. □

Actually, the inequality (2.4) holds for any $0 < q \leq p < \infty$. By Lemma 2.1 and Theorem 2.3, we build the key inequality for subsequent proof.

Corollary 2.4. *Let $p > 2$. Then, for any $f \in L_{\mathbb{A}}^2(0,T; L^p(\mathscr{C}))$, there is positive constant $C(p)$ such that*

$$\left\| \int_0^t f(s) dW_s \right\|_p \leq C(p) \left(\int_0^t \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}, \quad 0 \leq t \leq T. \tag{2.7}$$

Moreover, $L_{\mathbb{A}}^2(0,T; L^p(\mathscr{C})) \subseteq \mathcal{H}^p([0,T])$ and

$$\|f\|_{\mathcal{H}^p([0,t])} \leq \left(\int_0^t \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}, \quad 0 \leq t \leq T. \tag{2.8}$$

Proof. According to Theorem 2.3, for any $f \in L_{\mathbb{A}}^2(0,T; L^p(\mathscr{C}))$, one has

$$\left\| \left(\int_0^t |f(s)|^2 ds \right)^{\frac{1}{2}} \right\|_p \leq \left(\int_0^t \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}. \tag{2.9}$$

Since $\|f(s)\|_p = \|f(s)^*\|_p$ for any $0 \leq s \leq T$,

$$\|f\|_{\mathcal{H}^p([0,t])} = \max \left\{ \left\| \left(\int_0^t |f(s)|^2 ds \right)^{\frac{1}{2}} \right\|_p, \left\| \left(\int_0^t |f(s)^*|^2 ds \right)^{\frac{1}{2}} \right\|_p \right\} \leq \left(\int_0^t \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}.$$

Combining with (2.3), we have (2.7) immediately. □

Then, we state the parity of each element of $L^p(\mathscr{C})$. Let the parity operator P be automorphism map on von Neumann algebra $\mathscr{C}$ generated by bounded linear operators on $\Lambda(\mathscr{H})$ as is in [13, 15, 33].

Definition 2.4. For any $h \in L^p(\mathscr{C})$, h is said to be odd if $Ph = -h$, h is said to be even if $Ph = h$. And, h has definite parity if h is even or odd.

In fact, for any $1 < p < \infty$,

$$L^p(\mathscr{C}) = L^p(\mathscr{C}_o) \oplus L^p(\mathscr{C}_e), \tag{2.10}$$

where $L^p(\mathscr{C}_e)$, $L^p(\mathscr{C}_o)$ denote the even part and the odd part, respectively. Accurately speaking, for any $h \in L^p(\mathscr{C})$,

$$h = \frac{h + Ph}{2} + \frac{h - Ph}{2} = h_e + h_o,$$

where h_e and h_o are even and odd, respectively. Since P is isometric on $L^p(\mathscr{C})$,

$$\max\{\|h_o\|_p, \|h_e\|_p\} \le \|h\|_p \le \|h_o\|_p + \|h_e\|_p. \tag{2.11}$$

Let $\mathscr{E}$ denote the algebra of even polynomials in the fields $\{\Psi(u) : u \in \mathscr{H}\}$, and let $\mathscr{C}_e$ be W^*-subalgebra of $\mathscr{C}$ generated by $\mathscr{E}$. If $h \in L^p(\mathscr{C})$ is even there is a sequence $\{h_n\}$ in $\mathscr{E}$ such that $h_n \to h$ in $L^p(\mathscr{C})$, and therefore $h_n^* \to h^*$ in $L^p(\mathscr{C})$. It follows that h^* is also even. Similarly, if g is odd in $L^p(\mathscr{C})$, there is a sequence $\{g_n\}$ of odd polynomials in the fields with $g_n \to g$ and $g_n^* \to g^*$ in $L^p(\mathscr{C})$, that is, g^* is odd as well. In addition, if $h = h^*$ in $L^p(\mathscr{C})$ and $h = h_e + h_o$, then $h_e = h_e^*$ and $h_o = h_o^*$ in $L^p(\mathscr{C})$.

Lemma 2.5. *[13, Lemma 3.15] Let $\{W_t\}_{t \ge t_0}$ be a martingale adapted to the family $\{\mathscr{C}_t; t \in \mathbb{R}^+\}$. If $h \in L^p(\mathscr{C}_{t_0})$ has definite parity, then*

$$h(W_{t_2} - W_{t_1}) = \pm(W_{t_2} - W_{t_1})h, \ t_0 \le t_1 \le t_2$$

depending on whether h is even or odd.

Lemma 2.6. *[28, Theorem 1.8.2 Bihari inequality] Let $\rho : [0, +\infty) \to [0, +\infty)$ be a continuous and non-decreasing function vanishing at 0 satisfying $\int_{0^+} \frac{dr}{\rho(r)} = \infty$. Suppose $u(t)$ is a continuous nonnegative function on $[t_0, T]$ such that*

$$u(t) \le u_0 + \int_{t_0}^{t} \phi(r)\rho(u(r))dr, \ t_0 \le t \le T, \tag{2.12}$$

where u_0 is a nonnegative constant and $\phi : [t_0, T] \to \mathbb{R}^+$, then

$$u(t) \le U^{-1}\left(U(u_0) + \int_{t_0}^{t} \phi(r)dr\right), \quad t_0 \le t \le T,$$

where $U(t) = \int_{t_0}^{t} \frac{1}{\rho(r)}dr$, U^{-1} is the convex function of U. In particular, $u_0 = 0$, then $u(t) = 0$ for all $t_0 \le t \le T$.

Lemma 2.7. *[42, Lemma 2.5.14 Young's convolution inequality] Let $p \ge 1$. If $f \in L^p(\mathbb{R}^n)$, $g \in L^1(\mathbb{R}^n)$, then*

$$\|f * g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{L^p(\mathbb{R}^n)} \|g\|_{L^1(\mathbb{R}^n)}.$$

3 L^p-solutions of QSDEs in finite time horizon

In this section, we proved the existence and uniqueness of the L^p-solution with Osgood condition and the stability of the L^p-solution with Lipschitz condition of QSDEs in finite time horizon are proved for $p > 2$, respectively. The proof of Theorem 3.1 is similar to the classical methods [14, 18]. For the convenience of the reader, we give the proof in detail.

3.1 The existence and uniqueness of L^p-solutions of QSDEs

In this subsection, assumptions **(A1), (A2)** and **(A3')** of Assumption 1.1 hold. The existence and uniqueness of L^p-solutions of QSDE (1.3) and QSDE (1.6) are proved.

Theorem 3.1. *Under the assumptions of this subsection, QSDE* (1.6) *admits one and only one solution* $X_{(\cdot)} \in C_{\mathbb{A}}([0,T]; L^p(\mathscr{C}))$.

Proof. We shall deal with the existence and uniqueness separately.

Existence: The proof of the existence is divided into three steps.

Step 1. The iteration sequence $\{X_t^{(n)}\}_{n\geq 0}$ is well-defined for any $t_0 \leq t \leq T$. Let $T > t_0$, $t_0 \leq t \leq T$ be fixed. For any non-negative integer n, define $X_t^{(n)}$ in $L^p(\mathscr{C})$ inductively by

$$X_t^{(n+1)} = Z + \int_{t_0}^t F(X_s^{(n)}, s)dW_s + \int_{t_0}^t dW_s G(X_s^{(n)}, s) + \int_{t_0}^t H(X_s^{(n)}, s)ds. \tag{3.1}$$

Firstly, we claim that each $X_t^{(n)}$, $n \geq 1$, defines an adapted L^p-continuous process on $[t_0, T]$ by induction. By assumption **(A2)** of Assumption 1.1, $F(Z, s)$, $G(Z, s)$ and $H(Z, s)$ are L^p-continuous with respect to s and belong to $L^p(\mathscr{C}_s)$ for $t_0 \leq s \leq T$, then $X_t^{(1)}$ is well-defined for $t_0 \leq t \leq T$. Furthermore, we can obtain the boundedness of $X_t^{(1)}$ by the continuity on compact sets and easily verify that $t \mapsto X_t^{(1)}$ is continuous: $[t_0, T] \to L^p(\mathscr{C})$.

Now, if $X_t^{(n)}$ is assumed to be adapted and continuous, then $F(X_t^{(n)}, t)$, $G(X_t^{(n)}, t)$ and $H(X_t^{(n)}, t)$ are adapted, L^p-continuous and bounded on $[t_0,\ T]$, thus $X_t^{(n+1)}$ is adapted. For any $t_0 \leq t_1,\ t_2 \leq T$, by assumptions **(A1)** and **(A2)** of Assumption 1.1, Lemma 2.1, Lemma 2.2 and Hölder inequality,

$$\begin{aligned}
&\|X_{t_1}^{(n+1)} - X_{t_2}^{(n+1)}\|_p \\
&\leq \left\|\int_{t_1}^{t_2} F(X_s^{(n)}, s)dW_s\right\|_p + \left\|\int_{t_1}^{t_2} dW_s G(X_s^{(n)}, s)\right\|_p + \left\|\int_{t_1}^{t_2} H(X_s^{(n)}, s)ds\right\|_p \\
&\leq C(p)\left(\int_{t_1}^{t_2} \|F(X_s^{(n)}, s)\|_p^2 ds\right)^{\frac{1}{2}} + C(p)\left(\int_{t_1}^{t_2} \|G(X_s^{(n)}, s)\|_p^2 ds\right)^{\frac{1}{2}} + C(T)\left(\int_{t_1}^{t_2} \|H(X_s^{(n)}, s)\|_p^2 ds\right)^{\frac{1}{2}}.
\end{aligned}$$

This implies that $t \mapsto X_t^{(n+1)}$ is L^p-continuous on $[t_0, T]$. Hence, we have proved our claim by induction.

Step 2. The sequence of iteration is convergent under the given conditions. For any $t_0 \leq t \leq T$, by Minkowski-type inequality,

$$\begin{aligned}
&\|X_t^{(n+1)} - X_t^{(n)}\|_p \\
&\quad \leq \left\| \int_{t_0}^t \{F(X_s^{(n)}, s) - F(X_s^{(n-1)}, s)\} dW_s \right\|_p + \left\| \int_{t_0}^t dW_s \{G(X_s^{(n)}, s) - G(X_s^{(n-1)}, s)\} \right\|_p \\
&\qquad + \left\| \int_{t_0}^t \{H(X_s^{(n)}, s) - H(X_s^{(n-1)}, s)\} ds \right\|_p .
\end{aligned}$$

By similar analysis as above, the elementary inequality $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, Hölder inequality and assumption **(A3')** of Assumption 1.1, there exists a constant C such that

$$\begin{aligned}
&\|X_t^{(n+1)} - X_t^{(n)}\|_p^2 \\
&\quad \leq 3C(p)^2 \int_{t_0}^t \|F(X_s^{(n)}, s) - F(X_s^{(n-1)}, s)\|_p^2 ds + 3C(p)^2 \int_{t_0}^t \|G(X_s^{(n)}, s) - G(X_s^{(n-1)}, s)\|_p^2 ds \\
&\qquad + 3C(T)^2 \int_{t_0}^t \|H(X_s^{(n)}, s) - H(X_s^{(n-1)}, s)\|_p^2 ds \\
&\quad \leq C(p,T) \int_{t_0}^t \Big(\|F(X_s^{(n)}, s) - F(X_s^{(n-1)}, s)\|_p^2 + \|G(X_s^{(n)}, s) - G(X_s^{(n-1)}, s)\|_p^2 \\
&\qquad + \|H(X_s^{(n)}, s) - H(X_s^{(n-1)}, s)\|_p^2 \Big) ds \\
&\quad \leq C(p,T) \int_{t_0}^t \rho\left(\|X_s^{(n)} - X_s^{(n-1)}\|_p^2 \right) ds,
\end{aligned}$$

where $C(p,T) = 3\max\{C(p)^2,\ C(T)^2\}$. Therefore, for any $n,\ k \geq 1,\ t_0 \leq t \leq T$,

$$\|X_t^{(n+k)} - X_t^{(n)}\|_p^2 \leq C(p,T) \int_{t_0}^t \rho\left(\|X_s^{(n+k-1)} - X_s^{(n-1)}\|_p^2 \right) ds.$$

Since each $X_t^{(n)}$ is L^p-continuous process on $[t_0, T]$ for any $n \in \mathbb{N}^+$, $\|X_t^{(n)}\|_p$ is uniformly bounded on $[t_0, T]$. Set

$$u_{n,k}(t) = \sup_{t \in [t_0, T]} \|X_t^{(n+k)} - X_t^{(n)}\|_p^2,$$

which is also uniformly bounded, then

$$u_{n,k}(t) \leq C(p,T) \int_{t_0}^t \rho(u_{n-1,k}(s)) ds.$$

Let $v_n(t) = \sup_k u_{n,k}(t)$, $t_0 \leq t \leq T$. Then,

$$0 \leq v_n(t) \leq C(p,T) \int_{t_0}^t \rho(v_{n-1}(s)) ds.$$

Denote

$$\alpha(t) = \lim_{n \to +\infty} \sup v_n(t), \quad t_0 \leq t \leq T.$$

Applying Lebesgue dominated convergence theorem, we get

$$0 \le \alpha(t) \le C(p,T) \int_{t_0}^{t} \rho(\alpha(s))ds, \quad t_0 \le t \le T.$$

Hence, by Lemma 2.6, one deduces

$$\alpha(t) = 0, \quad t_0 \le t \le T,$$

which implies that $\{X_t^{(n)}\}_{n\ge 0}$ is a Cauchy sequence in $L^p(\mathscr{C})$.

Step 3. $X_{(\cdot)} \in C_{\mathbb{A}}([t_0,T]; L^p(\mathscr{C}))$ is the solution of QSDE (1.6). Since $\{X_t^{(n)}\}_{n\ge 0}$ is a Cauchy sequence in $L^p(\mathscr{C})$, there exists $X_t \in L^p(\mathscr{C})$ such that for any $t_0 \le t \le T$

$$\lim_{n\to\infty} \|X_t^{(n)} - X_t\|_p = 0.$$

Then, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}
\|X_{t_1} - X_{t_2}\|_p &= \|X_{t_1} - X_{t_1}^{(n)} + X_{t_1}^{(n)} - X_{t_2}^{(n)} + X_{t_2}^{(n)} - X_{t_2}\|_p \\
&\le \|X_{t_1} - X_{t_1}^{(n)}\|_p + \|X_{t_1}^{(n)} - X_{t_2}^{(n)}\|_p + \|X_{t_2}^{(n)} - X_{t_2}\|_p \\
&< \varepsilon, \text{ as } n \to \infty, \ \forall\ t_1, t_2 \in [t_0, T] \ \text{ satisfying } |t_1 - t_2| < \delta.
\end{aligned}$$

It shows that X_t is L^p-continuous and adapted on $[t_0,T]$ since $X_t^{(n)}$ is L^p-continuous and adapted.

We shall prove that $\{X_t\}_{t\ge t_0}$ is the solution of

$$X_t = Z + \int_{t_0}^{t} F(X_s,s)dW_s + \int_{t_0}^{t} dW_s G(X_s,s) + \int_{t_0}^{t} H(X_s,s)ds, \text{ a.s. } t_0 \le t \le T.$$

Indeed,

$$\begin{aligned}
&\left\| \int_{t_0}^{t} F(X_s^{(n)},s)dW_s - \int_{t_0}^{t} F(X_s,s)dW_s \right\|_p^2 \\
&\le C(p)^2 \|F(X_s^{(n)},s) - F(X_s,s)\|^2_{\mathcal{H}^p([t_0,t])} \\
&\le C(p)^2 \int_{t_0}^{t} \|F(X_s^{(n)},s) - F(X_s,s)\|_p^2 ds \\
&\le C(p)^2 \int_{t_0}^{t} \rho\left(\|X_s^{(n)} - X_s\|_p^2\right) ds \\
&\to 0, \quad n \to \infty,
\end{aligned}$$

since $X_s^{(n)} \to X_s$ in $L^p(\mathscr{C})$ for any $t_0 \le s \le T$ and ρ is continuous. Similarly,

$$\int_{t_0}^{t} dW_s G(X_s^{(n)},s) \to \int_{t_0}^{t} dW_s G(X_s,s) \text{ and } \int_{t_0}^{t} H(X_s^{(n)},s)ds \to \int_{t_0}^{t} H(X_s,s)ds$$

in $L^p(\mathscr{C})$. Taking limits on both sides of (3.1), it deduces that

$$\begin{aligned}
X_t &= \lim_{n\to\infty} X_t^{(n+1)} \\
&= \lim_{n\to\infty} \left(Z + \int_{t_0}^{t} F(X_s^{(n)},s)dW_s + \int_{t_0}^{t} dW_s G(X_s^{(n)},s) + \int_{t_0}^{t} H(X_s^{(n)},s)ds \right) \\
&= Z + \int_{t_0}^{t} F(X_s,s)dW_s + \int_{t_0}^{t} dW_s G(X_s,s) + \int_{t_0}^{t} H(X_s,s)ds.
\end{aligned}$$

That is, $\{X_t\}_{t\geq t_0}$ is a solution for any $t_0 \leq t \leq T$.

Uniqueness: Suppose that $\{Y_t\}_{t\geq t_0}$ is another adapted L^p-continuous solution with $Y_{t_0} = Z$. Then, by QSDE (1.6), we obtain

$$Y_t = Z + \int_{t_0}^t F(Y_s, s)dW_s + \int_{t_0}^t dW_s G(Y_s, s) + \int_{t_0}^t H(X_s, s)ds, \text{ a.s. } t_0 \leq t \leq T.$$

Furthermore, for any $t_0 \leq t \leq T$,

$$\begin{aligned}\|X_t - Y_t\|_p \leq & \left\| \int_{t_0}^t \{F(X_s, s) - F(Y_s, s)\}dW_s \right\|_p + \left\| \int_{t_0}^t dW_s\{G(X_s, s) - G(Y_s, s)\} \right\|_p \\ & + \left\| \int_{t_0}^t \{H(X_s, s) - H(Y_s, s)\}ds \right\|_p .\end{aligned}$$

Continuing to use the same technique as **Step 2** of existence, we can yield that

$$\|X_t - Y_t\|_p^2 \leq C(p, T) \int_{t_0}^t \rho\left(\|X_s - Y_s\|_p^2\right) ds, \quad t_0 \leq t \leq T.$$

It follows that,

$$\|X_t - Y_t\|_p = 0, \text{ a.s. } t_0 \leq t \leq T,$$

that is, the solution is unique. The proof is finished. □

As described in [8], the Itô product rule $dA(\chi_{[0,t]})dA^*(\chi_{[0,t]}) = dt$ holds for any $t \geq 0$. Based on [16], let

$$\xi_t = \alpha_1 A(u\chi_{[0,t)}) + \alpha_2 A^*(u\chi_{[0,t)}), \quad t \geq 0, \ \alpha_1, \alpha_2 \in \mathbb{C},$$

the stochastic integral $\int_0^t f(s)d\xi_s$ defines a quantum martingale for any $f \in \mathcal{S}^p_{\mathbb{A}}(\mathbb{R}^+)$. Next, let $A_t := A(\chi_{[0,t)})$, we study the properties of the L^p-solutions of QSDE (1.3) with respect to the fermion creation process $A(t)$ and annihilation process $A(t)^*$ on the basis of martingale inequalities. From Lemma 2.5 and the canonical anti-commutation relation, we derive the corresponding noncommutative Burkholder-Gundy inequalities.

Theorem 3.2. *Let $f : [0, T] \to L^p(\mathscr{C})$ be adapted processes with $p \geq 2$. Then, for any $0 \leq t \leq T$, $\int_0^t f(s)dA_s$ and $\int_0^t dA^*_s f(s)$ are L^p-martingales and*

$$\begin{aligned}\left\| \int_0^t f(s)dA_s \right\|_p &\leq 2\beta_p \left(\int_0^t \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}, \\ \left\| \int_0^t dA^*_s f(s) \right\|_p &\leq 2\beta_p \left(\int_0^t \|f(s)\|_p^2 ds \right)^{\frac{1}{2}}.\end{aligned} \tag{3.2}$$

Proof. First, we consider simple adapted L^p-process $f \in \mathcal{S}^p_{\mathbb{A}}([0, T])$, then $\int_0^t f(s)dA_s$ and $\int_0^t dA^*_s f(s)$ are L^p-martingales.

Let

$$0 = t_0 \leq t_1 \leq t_2 \leq \ldots \leq t_n = t$$

be a partition of $[0,t]$. Then,

$$Q(t)=\int_0^t f(s)dA_s=\sum_{k=0}^{n-1} f(t_k)\left(A_{t_{k+1}}-A_{t_k}\right),$$

$$Q(t_k)=\sum_{i=0}^{k-1} f(t_i)\left(A_{t_{i+1}}-A_{t_i}\right),\quad k\in\mathbb{Z},\ 0\le k\le n.$$

Define the martingale difference of $Q(t)$ as

$$dQ_k=Q(t_{k+1})-Q(t_k)=f(t_k)(A_{t_{k+1}}-A_{t_k}),\quad k\in\mathbb{Z}.$$

By Theorem 2.1 of [33], there exists β_p such that

$$\left\|\int_0^t f(s)dA_s\right\|_p\le\beta_p\max\left\{\left\|\left(\sum_{k\ge0}|dQ_k|^2\right)^{\frac12}\right\|_p,\quad\left\|\left(\sum_{k\ge0}|dQ_k^*|^2\right)^{\frac12}\right\|_p\right\}.\tag{3.3}$$

By the canonical anticommutation relation:

$$A_tA_t^*+A_t^*A_t=t,\quad t\ge0,$$

one has

$$(A_t-A_s)(A_t^*-A_s^*)\le t-s,\quad (A_t^*-A_s^*)(A_t-A_s)\le t-s,\quad 0\le s\le t.\tag{3.4}$$

By means of (2.10), $f=f_e+f_o$ for any $f\in L^p(\mathscr{C})$, so

$$\begin{aligned}\sum_{k\ge0}|dQ_k|^2&=\sum_{k=0}^{n-1}\left(A_{t_{k+1}}^*-A_{t_k}^*\right)f(t_k)^*f(t_k)\left(A_{t_{k+1}}-A_{t_k}\right)\\&=\sum_{k=0}^{n-1}\left(f_e(t_k)^*-f_o(t_k)^*\right)\left(f(t_k)-f_o(t_k)\right)\left(A_{t_{k+1}}^*-A_{t_k}^*\right)\left(A_{t_{k+1}}-A_{t_k}\right)\\&\le\sum_{k=0}^{n-1}\left(f_e(t_k)^*-f_o(t_k)^*\right)\left(f(t_k)-f_o(t_k)\right)(t_{k+1}-t_k)\\&=\int_0^t|f_e(s)-f_o(s)|^2ds,\end{aligned}\tag{3.5}$$

and

$$\begin{aligned}\sum_{k\ge0}|dQ_k^*|^2&=\sum_{k=0}^{n-1}f(t_k)(A_{t_{k+1}}-A_{t_k})(A_{t_{k+1}}^*-A_{t_k}^*)f^*(t_k)\\&\le\sum_{k=0}^{n-1}f(t_k)f(t_k)^*(t_{k+1}-t_k)\\&=\int_0^t|f(s)^*|^2ds,\end{aligned}\tag{3.6}$$

where the above two inequalities are based on Lemma 2.5 and (3.4).

Substituting (3.5) and (3.6) into the right side of (3.3) and applying Theorem 2.3, we get

$$\begin{aligned}\left\|\int_0^t f(s)dA_s\right\|_p &\leq \beta_p \max\left\{\left\|\left(\int_0^t |f(s)^*|^2 ds\right)^{\frac{1}{2}}\right\|_p, \left\|\left(\int_0^t |f_e(s)-f_o(s)|^2 ds\right)^{\frac{1}{2}}\right\|_p\right\} \\ &\leq \beta_p \max\left\{\left(\int_0^t \|f(s)^*\|_p^2 ds\right)^{\frac{1}{2}}, \left(\int_0^t \|f_e(s)-f_o(s)\|_p^2 ds\right)^{\frac{1}{2}}\right\}.\end{aligned}$$

From (2.11),

$$\left\|\int_0^t f(s)dA_s\right\|_p \leq 2\beta_p \left(\int_0^t \|f(s)\|_p^2 ds\right)^{\frac{1}{2}}, \quad 0 \leq t \leq T. \tag{3.7}$$

Similarly, one has

$$\left\|\int_0^t dA_s^* f(s)\right\|_p \leq 2\beta_p \left(\int_0^t \|f(s)\|_p^2 ds\right)^{\frac{1}{2}}, \quad 0 \leq t \leq T. \tag{3.8}$$

Finally, (3.2) can be obtained by (3.7) and (3.8) directly since the general adapted L^p-processes can be approximated by simple processes. □

As an immediate consequence of Theorem 3.1 and Theorem 3.2, we have the following result.

Corollary 3.3. *Under the assumptions of this subsection, there is a unique solution $M_{(\cdot)} \in C_{\mathbb{A}}([t_0, T]; L^p(\mathscr{C}))$ of QSDE (1.3) with initial condition $M_{t_0} = Z \in L^p(\mathscr{C}_{t_0})$ on $[t_0, T]$.*

If Lipschitz condition **(A3)** substitutes for Osgood condition **(A3')** in Theorem 3.1 and Corollary 3.3, then the existence and uniqueness of L^p-solution of QSDE (1.3) and QSDE (1.6) are obvious.

3.2 The stability of L^p-solutions of QSDEs

In this subsection, we shall prove that the solutions of QSDE (1.3) and QSDE (1.6) are stable, namely, small changes in the initial condition Z and in the coefficients F, G and H with Lipschitz condition lead to small changes in the solutions over a given finite time horizon $[t_0, T]$ when assumptions **(A1), (A2), (A3)** of Assumption 1.1 hold.

Let $\{X_t\}_{t\geq t_0}$, $\{Y_t\}_{t\geq t_0}$ be the solution of QSDE (1.6) with initial conditions $X_{t_0} = Z$ and $Y_{t_0} = Z'$ for any X_{t_0}, $Y_{t_0} \in L^p(\mathscr{C}_{t_0})$, respectively. The solution X_t is stable under the changes in the initial condition as follows:

Theorem 3.4. *Given that the above conditions hold, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $\|Z - Z'\|_p < \delta$, then $\|X_t - Y_t\|_p < \varepsilon$ for all $t_0 \leq t \leq T$.*

Proof. Similar to the analysis Theorem 3.1,

$$\begin{aligned}\|X_t^{(n+1)}-Y_t^{(n+1)}\|_p^2 \leq & 4\|Z-Z'\|_p^2+4C(p,T)\Big(\int_{t_0}^t \|F(X_s^{(n)})-F(Y_s^{(n)})\|_p^2 ds \\ & +\int_{t_0}^t \|G(X_s^{(n)})-G(Y_s^{(n)})\|_p^2 ds+\int_{t_0}^t \|H(X_s^{(n)})-H(Y_s^{(n)})\|_p^2 ds\Big) \\ \leq & 4\|Z-Z'\|_p^2+4C(p,T)\int_{t_0}^t \|X_s^{(n)}-Y_s^{(n)}\|_p^2 ds.\end{aligned}$$

Since $\lim\limits_{\varepsilon\to 0}\|Z-Z'\|_p=0$, by Lemma 2.6,

$$\lim_{\varepsilon\to 0}\|X_t^{(n+1)}-Y_t^{(n+1)}\|_p^2=0,\ \text{a.s. } t_0\leq t\leq T.$$

Setting $n\to\infty$, we conclude that

$$\lim_{\varepsilon\to 0}\|X_t-Y_t\|_p^2=0,\ \text{a.s. } t_0\leq t\leq T.$$

The result is our desired. $\square$

In a similar manner, we can get the convergence theorem.

Theorem 3.5. *Let $F,\ G,\ H,\ F_n,\ G_n,\ H_n$, for $n=1,\ 2,\cdots$, satisfy assumptions **(A1), (A2)** and **(A3)** of Assumption 1.1, and W_t be as in (1.5). Assume that $F_n\to F,\ G_n\to G,\ H_n\to H$ in $L^p(\mathscr{C})$ as $n\to\infty$, uniformly on $L^p(\mathscr{C})\times[t_0,T]$, and the initial condition $Z_n\to Z$ in $L^p(\mathscr{C}_{t_0})$. Then $X_n(t)\to X(t)$ in $L^p(\mathscr{C})$ uniformly on compact set $[t_0,T]$, where $X(t)$ is the solution of the QSDE*

$$\begin{cases} dX(t)=F(X(t),t)dW_t+dW_tG(X(t),t)+H(X(t),t)dt,\ \text{in } [t_0,T],\\ X(t_0)=Z,\end{cases}$$

and $X_n(t)$ is the corresponding solution with F,G,H,Z replaced by F_n,G_n,H_n,Z_n respectively.

Likewise, by Lemma 3.2 and Corollary 3.3, we can obtain the results related to QSDE (1.3).

Corollary 3.6. *With the above assumptions, the L^p-solution $M_{(\cdot)}\in C_{\mathbb{A}}([0,T];L^p(\mathscr{C})$ of QSDE (1.3) is stable on $[t_0,T]$ when initial condition $M_{t_0}=Z\in L^p(\mathscr{C}_{t_0})$ and the coefficients change slightly, respectively.*

In this section, the initial condition X_{t_0} of QSDE (1.6) is replace by $X_{t_0}=Z+R(X)$ where $R(\cdot):L^p(\mathscr{C})\to L^p(\mathscr{C})$ constitutes the nonlocal condition. That is,

$$\begin{cases} dX_t=F(X_t,t)dW_t+dW_tG(X_t,t)+H(X_t,t)dt,\ \text{in } [t_0,T],\\ X_{t_0}=Z+R(X).\end{cases}\tag{3.9}$$

Furthermore, R is continuous and adapted and there exists a constant $0<C(R)<1$ such that

$$\|R(x_1)-R(x_2)\|_p\leq C(R)\|x_1-x_2\|_p,\quad \forall\ x_1,\ x_2\in L^p(\mathscr{C}).\tag{3.10}$$

Theorem 3.1, Theorem 3.4 and Theorem 3.5 hold. Similarly, Corollary 3.3 and Corollary 3.6 also hold if nonlocal condition $M_{t_0}=Z+R(M)$ of QSDE (1.3) replaces of the initial condition $M_{t_0}=Z$.

4 L^p-solutions of QSDEs in infinite time horizon

Based on the analysis of Section 3, this section is devoted to proving the existence and uniqueness of L^p-solution and the dependence of L^p-solutions on initial value and coefficients of QSDE (1.6) for $p > 2$ whose integral form is

$$X_t = X_{t_0} + \int_{t_0}^{t} F(X_s, s)dW_s + \int_{t_0}^{t} dW_s G(X_s, s) + \int_{t_0}^{t} H(X_s, s)ds,\ t \geq t_0, \tag{4.1}$$

provided that assumptions **(A1), (A2), (A3)** and **(A4)** of Assumption 1.1 hold in infinite time horizon.

Firstly, we introduce the special linear spaces of infinite time horizon whose values belong to noncommutative space $L^p(\mathscr{C})$. Let $1 \leq p, q < \infty$ and $\mu \in \mathbb{R}$ be fixed. Define

$$L^{q,\mu}(0,\infty; L^p(\mathscr{C})) := \left\{ f : [0,\infty) \to L^p(\mathscr{C}) \middle| f \text{ is measurable}, \int_0^\infty e^{q\mu t} \|f(t)\|_p^q dt < \infty \right\},$$

and

$$L_{\mathbb{A}}^{q,\mu}(0,\infty; L^p(\mathscr{C})) := \left\{ f : [0,\infty) \to L^p(\mathscr{C}) \middle| f \in L^{q,\mu}(0,\infty; L^p(\mathscr{C})) \text{ and } f(\cdot) \text{ is adapted} \right\}.$$

Lemma 4.1. *Fix $\mu \in \mathbb{R}$ with $\frac{K_H}{\mu} + \frac{(K_F+K_G)\beta_p}{\sqrt{2\mu}} < \infty$. Then for any $X_t,\ X_t' \in L_{\mathbb{A}}^{2,-\mu}(t_0,\infty; L^p(\mathscr{C}))$, it holds that*

$$\left(\int_{t_0}^{\infty} e^{-2\mu t} \left\| \int_{t_0}^{t} \{F(X_s, s) - F(X_s', s)\}dW_s \right\|_p^2 dt \right)^{\frac{1}{2}} \leq \frac{\beta_p K_F}{\sqrt{2\mu}} \left(\int_{t_0}^{\infty} e^{-2\mu t} \|X_t - X_t'\|_p^2 dt \right)^{\frac{1}{2}}, \tag{4.2}$$

$$\left(\int_{t_0}^{\infty} e^{-2\mu t} \left\| \int_{t_0}^{t} dW_s\{G(X_s, s) - G(X_s', s)\} \right\|_p^2 dt \right)^{\frac{1}{2}} \leq \frac{\beta_p K_G}{\sqrt{2\mu}} \left(\int_{t_0}^{\infty} e^{-2\mu t} \|X_t - X_t'\|_p^2 dt \right)^{\frac{1}{2}}, \tag{4.3}$$

and

$$\left(\int_{t_0}^{\infty} e^{-2\mu t} \left\| \int_{t_0}^{t} \{H(X_s, s) - H(X_s', s)\}ds \right\|_p^2 dt \right)^{\frac{1}{2}} \leq \frac{K_H}{\mu} \left(\int_{t_0}^{\infty} e^{-2\mu t} \|X_t - X_t'\|_p^2 dt \right)^{\frac{1}{2}}. \tag{4.4}$$

Proof. By (2.7), one has

$$\left\| \int_{t_0}^{t} \{F(X_s, s) - F(X_s', s)\}dW_s \right\|_p \leq \beta_p \left(\int_{t_0}^{t} \|F(X_s, s) - F(X_s', s)\|_p^2 ds \right)^{\frac{1}{2}},\ t \geq 0.$$

Combining with Lemma 2.7 and assumption **(A3)** of Assumption 1.1, we have

$$
\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\left\|\int_{t_0}^{t}\{F(X_s,s)-F(X_s',s)\}dW_s\right\|_p^2 dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_{t_0}^{\infty} e^{-2\mu t}\beta_p^2\int_{t_0}^{t}\|F(X_s,s)-F(X_s',s)\|_p^2 ds dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_{t_0}^{\infty} e^{-2\mu t}\beta_p^2 K_F^2\int_{t_0}^{t}\|X_s-X_s'\|_p^2 ds dt\right)^{\frac{1}{2}}\\
&= \left(\beta_p^2 K_F^2\int_{t_0}^{\infty}\int_{t_0}^{t} e^{-2\mu(t-s)}e^{-2\mu s}\|X_s-X_s'\|_p^2 ds dt\right)^{\frac{1}{2}}\\
&\leq \beta_p K_F\left(\int_{t_0}^{\infty} e^{-2\mu t}dt\int_{t_0}^{\infty} e^{-2\mu t}\|X_t-X_t'\|_p^2 dt\right)^{\frac{1}{2}}\\
&= \frac{e^{-\mu t_0}\beta_p K_F}{\sqrt{2\mu}}\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t-X_t'\|_p^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

Similarly, by Theorem 2.3, Lemma 2.2 and Lemma 2.7,

$$
\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\left\|\int_{t_0}^{t}dW_s\{G(X_s,s)-G(X_s',s)\}\right\|_p^2 dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_{t_0}^{\infty} e^{-2\mu t}\beta_p^2\int_{t_0}^{t}\|G(X_s,s)-G(X_s',s)\|_p^2 ds dt\right)^{\frac{1}{2}}\\
&\leq \frac{e^{-\mu t_0}\beta_p K_G}{\sqrt{2\mu}}\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t-X_t'\|_p^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

From assumption **(A3)** of Assumption 1.1, Hölder inequality and Lemma 2.7, we get

$$
\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\left\|\int_{t_0}^{t}\{H(X_s,s)-H(X_s',s)\}ds\right\|_p^2 dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_{t_0}^{\infty} e^{-2\mu t}\left(\int_{t_0}^{t}\|H(X_s,s)-H(X_s',s)\|_p ds\right)^2 dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_{t_0}^{\infty} e^{-2\mu t}K_H^2\left(\int_{t_0}^{t}\|X_s-X_s'\|_p ds\right)^2 dt\right)^{\frac{1}{2}}\\
&= \left(\int_{t_0}^{\infty} K_H^2\int_{t_0}^{t} e^{-\mu(t-s)}e^{-\mu s}\|X_s-X_s'\|_p^2 ds dt\right)^{\frac{1}{2}}\\
&\leq \left(\int_{t_0}^{\infty} e^{-\mu t}dt\right)\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t-X_t'\|_p^2 dt\right)^{\frac{1}{2}}\\
&= \frac{e^{-\mu t_0}K_H}{\mu}\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t-X_t'\|_p^2 dt\right)^{\frac{1}{2}}.
\end{aligned}
$$

The proof is complete. □

On the basis of the above lemma, we define

$$\lambda_X := \inf\left\{\lambda \in \mathbb{R} \;\middle|\; e^{-\lambda t_0}\left(\frac{\beta_p(K_F+K_G)}{\sqrt{2\lambda}} + \frac{K_H}{\lambda}\right) \leq 1\right\} \in \mathbb{R}. \tag{4.5}$$

We now give the existence and uniqueness of L^p-solution of QSDE (1.6) in infinite horizon time.

Theorem 4.2. *Let $\mu \in (\lambda_X, \infty)$. Then for any initial condition $X_{t_0} \in L^p(\mathscr{C})$, QSDE (1.6) admits a unique adapted solution $X_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$. Further, the following estimate holds*

$$\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t\|_p^2 dt\right)^{\frac{1}{2}} \leq C_\mu \frac{1}{\sqrt{2\mu}}\|X_{t_0}\|_p, \tag{4.6}$$

where

$$C_\mu := \frac{1}{e^{\mu t_0} - \frac{K_H}{\mu} - \frac{\beta_p(K_F+K_G)}{\sqrt{2\mu}}}.$$

Proof. Let us prove the existence and uniqueness of the solution by Banach's fixed-point theorem. Firstly, we define a map $\Psi : L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C})) \to L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$ by

$$\Psi(X)(t) := X_{t_0} + \int_{t_0}^{t} F(X_s, s)dW_s + \int_{t_0}^{t} dW_s G(X_s, s) + \int_{t_0}^{t} H(X_s, s)ds, \;\; t \geq t_0,$$

for each $X_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$. By Lemma 4.1, the map Ψ is well-defined. Let $X'_{(\cdot)}, X_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$ be fixed. Then

$$\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\|\Psi(X)(t) - \Psi(X')(t)\|_p^2 dt\right)^{\frac{1}{2}} \\
&\leq \left(\int_{t_0}^{\infty} e^{-2\mu t}\left\|\int_{t_0}^{t}\{F(X_s, s) - F(X_s^{'}, s)\}dW_s\right\|_p^2 dt\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{t_0}^{\infty} e^{-2\mu t}\left\|\int_{t_0}^{t} dW_s\{G(X_s, s) - G(X_s^{'}, s)\}\right\|_p^2 dt\right)^{\frac{1}{2}} \\
&\quad + \left(\int_{t_0}^{\infty} e^{-2\mu t}\left\|\int_{t_0}^{t}\{H(X_s, s) - H(X_s^{'}, s)\}ds\right\|_p^2 dt\right)^{\frac{1}{2}} \\
&\leq e^{-\mu t_0}\left(\frac{\beta_p(K_F+K_G)}{\sqrt{2\mu}} + \frac{K_H}{\mu}\right)\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t - X_t^{'}\|_p^2 dt\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\mu \in (\lambda_X, \infty)$ with $\lambda_X \in \mathbb{R}$ defined by (4.7), it holds that $e^{-\mu t_0}\frac{\beta_p(K_F+K_G)}{\sqrt{2\mu}} + \frac{K_H}{\mu} < 1$. It shows that Ψ is a contraction map on the Banach space $L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$. Hence, there is a unique fixed point $X_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$ of Ψ, which is the solution of the equation (4.1). By the same calculation, we can deduce that

$$\left(1 - e^{-\mu t_0}\left(\frac{\beta_p(K_F+K_G)}{\sqrt{2\mu}} + \frac{K_H}{\mu}\right)\right)\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X_t\|_p^2 dt\right)^{\frac{1}{2}} \leq \|X_{t_0}\|_p\left(\int_{t_0}^{\infty} e^{-2\mu t}dt\right)^{\frac{1}{2}},$$

which implies (4.6). □

Based on Theorem 3.2 and Lemma 4.1, let

$$\lambda_M := \inf\left\{\lambda \in \mathbb{R} \;\middle|\; e^{-\lambda t_0}\left(\frac{2\beta_p(K_F+K_G)}{\sqrt{2\lambda}} + \frac{K_H}{\lambda}\right) \leq 1\right\} \in \mathbb{R}. \tag{4.7}$$

Corollary 4.3. *Given that the above assumptions hold, $\mu \in (\lambda_M, \infty)$ and initial condition $M_{t_0} \in L^p(\mathscr{C})$, QSDE (1.3) admits a unique adapted solution $M_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$. Furthermore, the following estimate holds*

$$\left(\int_{t_0}^{\infty} e^{-2\mu t}\|M_t\|_p^2 dt\right)^{\frac{1}{2}} \leq C_\mu \frac{1}{\sqrt{2\mu}}\|M_{t_0}\|_p, \tag{4.8}$$

where

$$C_\mu := \frac{1}{e^{\mu t_0} - \frac{K_H}{\mu} - \frac{2\beta_p(K_F+K_G)}{\sqrt{2\mu}}}.$$

Meanwhile, we get the dependence of L^p-solutions on initial value and coefficients.

Theorem 4.4. *Let F', G' and H' satisfy assumption as above, $\mu \in (\lambda_X, \infty)$ and X'_{t_0} be given. If $X'_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$ is the solution of QSDE (1.6) corresponding to (X'_{t_0}, F', G', H'), then*

$$\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\|X'_t - X_t\|_p^2 dt\right)^{\frac{1}{2}} \\
&\leq C_\mu \Big(\int_{t_0}^{\infty} e^{-2\mu t}\Big(\|X'_{t_0} - X_{t_0}\|_p + \left\|\int_{t_0}^{t}\{F'(X'_s, s) - F(X_s, s)\}dW_s\right\|_p \\
&\quad + \left\|\int_{t_0}^{t} dW_s\{G'(X'_s, s) - G(X_s, s)\}\right\|_p + \left\|\int_{t_0}^{t}\{H'(X'_s, s) - H(X_s, s)\}ds\right\|_p\Big)^2 dt\Big)^{\frac{1}{2}}.
\end{aligned} \tag{4.9}$$

Proof. Since $X'_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0, \infty; L^p(\mathscr{C}))$ is the solution of the equation (4.1) corresponding to $(X'_{t_0},\ F',\ G',\ H')$,

$$X'_t = X'_{t_0} + \int_{t_0}^{t} F'(X'_s, s)dW_s + \int_{t_0}^{t} dW_s G'(X'_s, s) + \int_{t_0}^{t} H'(X'_s, s)ds,\ t \geq t_0. \tag{4.10}$$

Let $\overline{X}_{(\cdot)} = X_{(\cdot)} - X'_{(\cdot)}$. From (4.1) and (4.10),

$$\overline{X}_t = \overline{X}_{t_0} + \int_{t_0}^{t} F(\overline{X}_s, s)dW_s + \int_{t_0}^{t} dW_s G(\overline{X}_s, s) + \int_{t_0}^{t} H(\overline{X}_s, s)ds,\ t \geq t_0,$$

where

$$\begin{aligned}
\overline{X}_{t_0} =& X'_{t_0} - X_{t_0} + \int_{t_0}^{t}\{F'(X'_s, s) - F(X'_s, s)\}dW_s \\
&+ \int_{t_0}^{t} dW_s\{G'(X'_s, s) - G(X'_s, s)\} + \int_{t_0}^{t}\{H'(X'_s, s) - H(X'_s, s)\}ds,
\end{aligned}$$

$$F(\overline{X}_s, s) = F(X'_s, s) - F(X_s, s), \quad G(\overline{X}_s, s) = G(X'_s, s) - G(X_s, s),$$

$$H(\overline{X}_s, s) = H(X'_s, s) - H(X_s, s).$$

By the same calculation as (4.6), we can obtain

$$
\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\|\overline{X}_t\|_p^2 dt\right)^{\frac{1}{2}} \\
&\leq C_\mu\left(\int_{t_0}^{\infty} e^{-2\mu t}\|\overline{X}_{t_0}\|_p^2 dt\right)^{\frac{1}{2}} \\
&\leq C_\mu\Big(\int_{t_0}^{\infty} e^{-2\mu t}\Big(\|X'_{t_0}-X_{t_0}\|_p + \Big\|\int_{t_0}^{t}\{F'(X'_s,s)-F(X'_s,s)\}dW_s\Big\|_p \\
&\qquad + \Big\|\int_{t_0}^{t} dW_s\{G'(X'_s,s)-G(X'_s,s)\}\Big\|_p + \Big\|\int_{t_0}^{t}\{H'(X'_s,s)-H(X'_s,s)\}ds\Big\|_p\Big)^2 dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

The proof is complete. □

Corollary 4.5. *Under the Corollary 4.3, let F', G' and H' satisfy assumptions as above, and M'_{t_0} be given. If $M'_{(\cdot)} \in L^{2,-\mu}_{\mathbb{A}}(t_0,\infty;L^p(\mathscr{C}))$ is the solution of QSDE (1.3) in infinite time horizon corresponding to (M'_{t_0}, F', G', H'), then*

$$
\begin{aligned}
&\left(\int_{t_0}^{\infty} e^{-2\mu t}\|M'_t - M_t\|_p^2 dt\right)^{\frac{1}{2}} \\
&\leq C_\mu\Big(\int_{t_0}^{\infty} e^{-2\mu t}\Big(\|M'_{t_0}-M_{t_0}\|_p + \Big\|\int_{t_0}^{t}\{M'(X'_s,s)-M(X_s,s)\}dA_s\Big\|_p \\
&\qquad + \Big\|\int_{t_0}^{t} dA_s^*\{G'(M'_s,s)-G(M_s,s)\}\Big\|_p + \Big\|\int_{t_0}^{t}\{H'(M'_s,s)-H(M_s,s)\}ds\Big\|_p\Big)^2 dt\Big)^{\frac{1}{2}}.
\end{aligned}
$$

5 The Self-adjointness and Markov Property

In this section, we discuss the self-adjointness and Markov property of L^p-solutions of QSDE (1.6) for $p > 2$. In order to prove the self-adjointness of L^p-solution, together with the description of parity in Section 2, we first give an auxiliary lemma.

Lemma 5.1. *Let $F : [0,T] \to L^p(\mathscr{C})$ be adapted and satisfy $\int_0^t \|F(s)\|_p^2 ds < \infty$. Suppose further that $F(t) = F(t)^* \in L^p(\mathscr{C}_e)$ for each $0 \leq t \leq T$. Then $\int_0^t F(s)dW_s$ is self-adjoint element of $L^p(\mathscr{C})$, and $\int_0^t F(s)dW_s = \int_0^t dW_s F(s)$.*

Proof. It is sufficient to consider the case that $F(t)$ is simple with values in $\mathscr{E}$ for any $0 \leq t \leq T$. Since F is simple, $F(t) = \sum_{k=0}^{n-1} F(t_k)\chi_{[t_k,t_{k+1})}(t)$ and

$$
\int_0^t F(s)dW_s = \sum_{k=0}^{n-1} F(t_k)(W_{t_{k+1}} - W_{t_k}),
$$

where $\{t_k\}_{k=0}^n$ is a partition of interval $[0,t]$.

On the other hand, $F(t)$ and W_t are hermitian for any $t \geq 0$, then

$$\begin{aligned}\left(\int_0^t F(s)dW_s\right)^* &= \left(\lim_{n\to\infty}\sum_{k=0}^{n-1} F(t_k)(W_{t_{k+1}} - W_{t_k})\right)^* \\ &= \sum_{k=0}^{n-1}\left(W_{t_{k+1}} - W_{t_k}\right)^* F(t_k)^* \\ &= \sum_{k=0}^{n-1}\left(W_{t_{k+1}} - W_{t_k}\right) F(t_k) \\ &= \int_0^t dW_s F(s).\end{aligned}$$

Since $F(t)$ is even for any $0 \leq t \leq T$, by Lemma 2.5,

$$\left(W_{t_{k+1}} - W_{t_k}\right) F(t_k) = F(t_k)\left(W_{t_{k+1}} - W_{t_k}\right), \quad \forall\ F(t_k) \in L^p(\mathscr{C}_e).$$

Then, $\int_0^t dW_s F(s) = \int_0^t F(s)dW_s$. It is obvious that $\int_0^t F(s)dW_s$ is self-adjoint element in $L^p(\mathscr{C})$ by virtue of $\int_0^t \|F(s)\|_p^2 ds < \infty$ and Corollary 2.4. The proof is complete. $\square$

Let $L^p(\mathscr{C})_{sa}$ denote the self-adjoint part of $L^p(\mathscr{C})$. Let $F_i,\ G_i : L^p(\mathscr{C})_{sa} \to L^p(\mathscr{C}),\ (i = 1, 2)$ be adapted and each F_i be an even function. Set

$$\widetilde{F}_i(h) = F_i(h_o), \quad \widetilde{G}_i(h) = G(h_e), \quad \forall\ h \in L^p(\mathscr{C})_{sa}.$$

Evidently, $\widetilde{F}_i(h),\ \widetilde{G}_i(h)\ (i = 1, 2)$ are even by Lemma 4.1 of [14] for any $h \in L^p(\mathscr{C})_{sa}$. Let

$$\widetilde{\Phi}_i := \widetilde{F}_i + \widetilde{G}_i, \quad i = 1, 2.$$

It is seen that $\widetilde{\Phi}_i$ satisfies the Osgood conditions and maps self-adjoint elements of $L^p(\mathscr{C})$ into self-adjoint elements of $L^p(\mathscr{C}_e)$. Then, we have the self-adjointness of the solutions of QSDEs.

Theorem 5.2. *Let $\widetilde{\Phi}_1,\ \widetilde{\Phi}_2$ be as above. Let $\widetilde{H} : L^p(\mathscr{C})_{sa} \to L^p(\mathscr{C})$ be adapted and satisfy Osgood conditions on $L^p(\mathscr{C})$ in Assumption 1.1. Thus, for any $Z = Z^*$, there is a unique self-adjoint, adapted, L^p-continuous solution $\{X_t\}_{t\geq t_0}$ of the following QSDE*

$$dX_t = \widetilde{\Phi}_1(X_t)dW_t + dW_t\widetilde{\Phi}_2(X_t) + \widetilde{H}(X_t)dt,\ t \geq t_0 \tag{5.1}$$

with initial condition $X_{t_0} = Z$.

Proof. Since $\widetilde{\Phi}_1,\ \widetilde{\Phi}_2,\ \widetilde{H}$ satisfy the Osgood conditions **(A3')** as in Assumption 1.1, it follows from Theorem 3.1 that QSDE (5.1) admits a unique L^p-continuous, adapted solution $\{X_t\}_{t\geq t_0}$ such that

$$X_t = Z + \int_{t_0}^t \widetilde{\Phi}_1(X_s)dW_s + \int_{t_0}^t dW_s\widetilde{\Phi}_2(X_s) + \int_{t_0}^t \widetilde{H}(X_s)ds,\ \text{a.s. } t \geq t_0.$$

Next, it is enough to prove the self-adjointness of the solution of QSDE (5.1). We can define the following integral equation by inductively with $X_{t_0} = Z$,

$$X_t^{(n+1)} = Z + \int_{t_0}^t \widetilde{\Phi}_1(X_s^{(n)})dW_s + \int_{t_0}^t dW_s\widetilde{\Phi}_2(X_s^{(n)}) + \int_{t_0}^t \widetilde{H}(X_s^{(n)})ds. \tag{5.2}$$

To prove the self-adjointness of X_t, it is sufficient to show that $X_t^{(n+1)}$ is self-adjoint by induction for all $n \geq 0$. It is obvious that $X_t^{(1)}$ is self-adjoint since $Z = Z^*$. Assume that $X_t^{(n)}$ is self-adjoint, then $\widetilde{\Phi}_i(X_s^{(n)}) \in L^p(\mathscr{C}_e) \cap L^p(\mathscr{C}_s)_{sa}$. By Lemma 5.1, $\int_{t_0}^t \widetilde{\Phi}_1(X_s^{(n)})dW_s$ and $\int_{t_0}^t dW_s\widetilde{\Phi}_2(X_s^{(n)})$ are self-adjoint. In addition, $\int_0^t H(X_s^{(n)})ds$ is also self-adjoint. Hence, $X_t^{(n+1)}$ is self-adjoint. Then, the fact that X_t is self-adjoint for any $t \geq t_0$ is easily proved, and the proof by iteration is similar to Theorem 3.1. □

Theorem 5.2 also holds when the initial condition $X_{t_0} = Z$ is replaced by nonlocal condition $X_{t_0} = Z + R(X)$ as described in Section 3 .

This result of self-adjointness of L^p-solutions is the basis for the study of studying optimal control problems of QSDEs. The Markov property is another important property of stochastic processes. Accardi, Frigerio, Lewis and their co-authors [2, 3, 5–7] have done a series of works on Markov properties of quantum Markov processes. Next, we discuss Markov property of the solution of the QSDEs by the transition probability which is consistent with Theorem 2.2, Corollary 2.3 and Corollary 2.4 of [15].

For any interval $I \subseteq [t_0, \infty)$, let $\mathscr{A}_I$ denote the W^*-algebra generated by $\mathbb{1}$ and the solution X_t of QSDE (1.6) for $t \in I$, and write $\mathscr{A}_s$ for $\mathscr{A}_{[s,s]}$. Since the solution X_t is adapted, i.e. $X_t \in L^p(\mathscr{C}_t)$ for all $t \geq t_0$, it follows that $\mathscr{A}_I$ is a W^*-subalgebra of $\mathscr{C}_t$ whenever $I \subseteq [t_0, t]$. Let $\tilde{\mathscr{A}_I} = \mathscr{A}_I \vee \beta(\mathscr{A}_I)$ be the W^*-subalgebra of $\mathscr{C}$ generated by $\mathscr{A}_I$ and $\beta(\mathscr{A}_I)$. It is clear that $\beta(\tilde{\mathscr{A}_I}) = \tilde{\mathscr{A}}$ and $\tilde{\mathscr{A}_s} \subseteq \mathscr{C}_s$ for any $s \geq t_0$.

Next, we denote the algebra generated by field differences. Let $\mathscr{F}_s$ denote the W^*-subalgebra of $\mathscr{C}$ generated by the field differences $\{W_\tau - W_s : \ t_0 \leq s \leq \tau\}$, $\tilde{\mathscr{A}_s} \vee \mathscr{F}_s$ be the W^*-subalgebra of $\mathscr{C}$ generated by $\tilde{\mathscr{A}_s}$ and $\mathscr{F}_s$. Thus, $\beta(\tilde{\mathscr{A}_s} \vee \mathscr{F}_s) = \tilde{\mathscr{A}_s} \vee \mathscr{F}_s$. Then, we obtain the Markov property of the adapted solution $\{X_t\}_{t \geq t_0}$ of QSDE (1.6).

Theorem 5.3. *Let assumptions* $(A1), (A2)$ *and* $(A3)$ *of Assumption 1.1 hold and* $\{X_t\}_{t \geq t_0}$ *be an adapted, unique, continuous* L^p*-solution of QSDE* (1.6)*, then* $X_s \in L^p(\tilde{\mathscr{A}_s} \vee \mathscr{F}_s)$ *for all* $t_0 \leq s \leq t$*. Moreover, the process* $\{X_t\}_{t \geq t_0}$ *is a Markov process in the following sense: for any* $s \geq t_0$ *and* $f \in L^{\frac{p}{2}}(\tilde{\mathscr{A}}_{[s,\infty)})$*, one has*

$$m(f|\tilde{\mathscr{A}}_{[t_0,s]}) = m(f|\tilde{\mathscr{A}_s}), \tag{5.3}$$

where $m(\cdot|\mathscr{B})$ *denotes the conditional expectation with respect to the subalgebra* $\mathscr{B}$ *of* $\mathscr{C}$*.*

The proof of Theorem 5.3 is similar to Theorem 2.2 of [15]. Furthermore, the result for the solution $\{M_t\}_{t \geq t_0}$ of QSDE (1.3) also holds.

6 Conclusion

In this paper, we have used some useful inequalities such as Hölder inequality, the non-commutative Burkholder-Gundy inequalities, Bihari inequality and Picard approximation to obtain the existence and uniqueness of solution of QSDEs driven by the fermion field in non-commutative space $L^p(\mathscr{C})$. Moreover, the stability, dependence on initial value and coefficients, self-adjointness and Markov property are developed for QSDEs. This paper will play a key role in studying optimal control problem of quantum control system for future work.

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