Existence and multiplicity of positive solutions of a fourth-order m-piont BVP with sign-changing Green's function

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Abstract

In this paper, we study the fourth-order m-point boundary value problem \begin{equation*} \left \{\begin{array}{lcr} u^{(4)}(t)=f(t,u(t)),\ t\in [0,1],\\ u'(0)=u''(0)=u(1)=0 , u''(1)-\sum^{m-2}_{i=1}\alpha_i u'''(\xi_i)=0, \end{array}\right. \end{equation*} with sign-changing Green's function. By using some fixed theorems and the properties of Green's function, we mainly establish the existence and multiplicity of positive solution for the problems under some suitable conditions.

EXISTENCE AND MULTIPLICITY OF POSITIVE SOLUTIONS OF A FOURTH-ORDER M-PIONT BVP WITH SIGN-CHANGING GREEN'S FUNCTION

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ABSTRACT. In this paper, we study the fourth-order m-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)),\ t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, u''(1) - \sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0, \end{cases}$$

with sign-changing Green's function. By using some fixed theorems and the properties of Green's function, we mainly establish the existence and multiplicity of positive solution for the problems under some suitable conditions.

1. Introduction

This paper is concerned with the solvability of the fourth-order m-point boundary value problem

$$\begin{cases} u^{(4)}(t) = f(t, u(t)),\ t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, u''(1) - \sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0, \end{cases} \tag{1.1}$$

where α_i, ξ_i and f satisfy the following assumptions

(H1) $\sum_{i=1}^{m-2} \alpha_i > \frac{10}{7}$ and $0 \leq \sum_{i=2}^{m-2} \alpha_i < \frac{1}{2}\Big(1 - \sqrt{\frac{5\sum_{i=1}^{m-2}\alpha_i - 4}{12\sum_{i=1}^{m-2}\alpha_i - 14}}\Big)$;

(H2) $\sqrt{\frac{5\sum_{i=1}^{m-2}\alpha_i - 4}{12\sum_{i=1}^{m-2}\alpha_i - 14}} < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1 - 2\sum_{i=2}^{m-2} \alpha_i$;

(H3)$f \in C([0,1] \times [0,+\infty),\ [0,+\infty))$, and satisfies

  (i) For each $u \in [0,+\infty)$, $f(t,u)$ is monotonically decreasing with respect to t;

  (ii) For each $t \in [0,1]$, $f(t,u)$ is monotonically increasing with respect to u.

To the best of our knowledge, the fourth-order boundary value problem models the deformations of an elastic beam in mechanics, so many authors pay more attention to studying the existence of solutions to this problem by using the variational method [20], the method of upper and lower solutions [4, 5, 19], iterative method[8], the fixed point index theory [7, 12, 13, 23], fixed point theorems [11, 14, 21] or the bifurcation theory[15, 16, 18]. A standard approach to studying positive solutions of the fourth-order boundary value problem such as (1.1) is to find the corresponding nonnegative Green's function $G(t,s)$ and seek solutions as fixed points of the

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integral operator, see [11, 23] and references therein. Recently, many authors analyzed some possibilities of finding positive solutions for second-order boundary value problems with the Dirichlet and periodic boundary conditions, for which the corresponding Green's functions change sign. We refer the readers to [6, 9, 10, 17, 22, 25] and references therein.

To our attention, Y. Zhang.etc [25] are concerned with the following fourth-order three-point BVP with sign-changing Green's function

$$\begin{cases} u^{(4)}(t) = f(t, u(t)),\ t \in [0, 1], \\ u'(0) = u''(0) = u'''(\eta) = u(1) = 0. \end{cases} \tag{1.2}$$

By imposing some suitable conditions on f and η, the authors obtained the existence of at least n-1 decreasing positive solutions of problem (1.2) by using the fixed point index theory. Furthermore, H. Djourdem.etc [9] studied the following fourth-order three-point BVP with sign-changing Green's function

$$\begin{cases} u^{(4)}(t) = f(t, u(t)),\ t \in [0, 1], \\ u'(0) = u''(0) = u(1) = 0, \alpha u''(1) - u'''(\eta) = 0. \end{cases} \tag{1.3}$$

By imposing some suitable conditions on f, α and η, the authors still obtained the existence of at least two positive and decreasing solutions of problem (1.3). by applying the two-fixed-point theorem due to Avery and Henderson.

Concerning the fourth-order BVP with sign-changing Green's function, as far as we know, no results for this more general fourth-order m-point boundary value problem (1.1). So we establish some new criterions for the existence of positive solutions for problem (1.1) by applying some fixed theorems, such as: Let K be a cone in a Banach space X. If D is a bounded open subset of X, we denote an open subset of K by $D_K = D \cap K$.

Definition 1.1. A map α is said to be a nonnegative continuous concave functional on a cone K of a real Banach space X if

$$\alpha : K \to [0, +\infty)$$

is continuous and

$$\alpha(\lambda u + (1 - \lambda)v) \geq \lambda\alpha(u) + (1 - \lambda)\alpha(v)$$

for all $u, v \in K$, $t \in [0, 1]$. Similarly, we say the map β is a nonnegative continuous convex functional on a cone K of a real Banach space X if

$$\beta : K \to [0, +\infty)$$

is continuous and

$$\beta(\lambda u + (1 - \lambda)v) \leq \lambda\beta(u) + (1 - \lambda)\beta(v)$$

for all $u, v \in K$, $t \in [0, 1]$.

Definition 1.2. Let ϕ, φ, α be a nonnegative, continuous functional on K. Then for positive real number a, b and c, we define some sets by

$$K(\phi, a) = \{x \in K : \phi(x) \leq a\},$$

$$K(\phi, \varphi; a, b) = \{x \in K : a \leq \varphi(x), \phi(x) \leq b\},$$

$$K(\phi, \varphi, \alpha; a, b, c) = \{x \in K : a \leq \alpha(x), \varphi(x) \leq b, \phi(x) \leq c\}.$$

Lemma 1.3.[1] Let D be an open bounded set with $D_K \neq \emptyset$ and $\overline{D_K} \neq K$. Assume that $T : \overline{D_K} \to K$ is a compact map such that $x \neq Tx$ for $x \in \partial D_K$. Then the fixed point index $i_K(T, D_K)$ has the following properties.

(1) If there exists $e \in K \backslash \{0\}$ such that $x \neq Tx + \lambda e$ for all $x \in \partial D_K$ and all $\lambda > 0$, then $i_K(T, D_K) = 0$.
(2) If $\|Tx\| \leq \|x\|$ for $x \in \partial D_K$, then $i_K(T, D_K) = 1$.
(3) Let D^1 be open in X with $\overline{D^1} \subset D_K$. If we have $i_K(T, D_K) = 1$ and $i_K(T, D_K^1) = 0$, then T has a fixed point in $D_K \backslash \overline{D_K^1}$. The same result holds if $i_K(T, D_K) = 0$ and $i_K(T, D_K^1) = 1$.

Lemma 1.4.[2] Let ψ and γ be increasing, nonnegative, and continuous functionals on K, and let ω be a nonnegative continuous functional on K with $\omega(0) = 0$ such that for some $a_3 > 0$ and $M > 0$,

$$\gamma(u) \leq \omega(u) \leq \psi(u), \|u\| \leq M\gamma(u)$$

for all $u \in K(\gamma, a_3)$. Suppose there exist a completely continuous operator $T : K(\gamma, a_3) \to K(\gamma, a_3)$ and $0 < a_1 < a_2 < a_3$ such that

$$\omega(\lambda u) \leq \lambda \omega(u) \ for \ 0 \leq \lambda \leq 1, \ u \in \partial K(\omega, a_2),$$

and
(1) $\gamma(Tu) > a_3$ for all $u \in \partial K(\gamma, a_3)$;
(2) $\omega(Tu) < a_2 a_3$ for all $u \in \partial K(\omega, a_2)$;
(3) $K(\psi, a_1) \neq \emptyset$ and $\psi(Tu) > a_1$ for all $u \in \partial K(\psi, a_1)$.
Then T has at least two fixed points u_1 and u_2 in $K(\gamma, a_3)$ such that

$$a_1 < \psi(u_1) \text{ with } \omega(u_1) < a_2,$$

$$a_2 < \omega(u_2) \text{ with } \gamma(u_2) < a_3.$$

Lemma 1.5.[3] Let K be a cone in a real Banach space X. Let ϕ and φ be nonnegative, continuous convex functional on K, α be a nonnegative, continuous concave functional on K, and ψ be a nonnegative, continuous functional on K satisfying $\psi(\lambda x) \leq \lambda \psi(x)$ for $0 \leq \lambda \leq 1$, such that for some positive numbers M and d, $\alpha(x) \leq \psi(x)$ and $\|x\| \leq M\phi(x)$, for all $x \in K(\phi, d)$. Suppose

$$T : K(\phi, d) \to K(\phi, d)$$

is completely continuous and there exist positive numbers a, b and c with $a < b$ such that
(i) $\{x \in K(\phi, \varphi, \alpha; b, c, d) : \alpha(x) > b\} \neq \emptyset$ and $\alpha(Tx) > b$ for $x \in K(\phi, \varphi, \alpha; b, c, d)$;
(ii) $\alpha(Tx) > b \ for \ x \in K(\phi, \alpha; b, d)$ with $\varphi(Tx) > c$;
(iii) $0 \notin K(\phi, \psi; a, d)$ and $\psi(Tx) < a$ with $x \in K(\phi, \psi; a, d)$ with $\psi(x) = a$.
Then T has at least three fixed points $x_1, x_2, x_3 \in K(\phi, d)$ such that

$$\phi(x_i) \leq d \ for \ i = 1, 2, 3;$$

$$b < \alpha(x_1), \ \ a < \psi(x_2) \text{ with } \alpha(x_2) < b$$

and

$$\psi(x_3) < a.$$

The rest of this article is organized in the following way: In Section 2, we state some preliminary results considering the Green's function may change sign. In Section 3, some new results concerning the positive solutions of a fourth-order m-piont BVP with sign-changing Green's function are proved. In Section 4, we give some examples to illustrate our main results.

2. **Preliminaries and lemmas**

Throughout this paper, let X denote the Banach space $C[0,1]$ with the norm $\|u\| = \max_{t\in[0,1]} |u(t)|$, for $u \in X$. In addition, define a subcone K_0 of X by

$$K_0 = \{u \in X : u(t) \text{ is nonnegative and decreasing on } [0,1]\}.$$

To discuss BVP(1.1), we consider the corresponding fourth-order linear boundary value problem (LBVP)

$$\begin{cases} u^{(4)}(t) = y(t), t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, u''(1) - \sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0. \end{cases} \tag{2.1}$$

Lemma 2.1. Assume that (H1) and (H2) hold. Then for any given $y(t) \in X$, the LBVP(2.1) has a unique solution $u(t) = \int_0^1 G(t,s)y(s)ds$, where

$$G(t,s) = \frac{1}{6}\begin{cases} (t-s)^3 - (1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}, \ s \le t, s \le \xi_1, \\ -(1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}, \ t \le s \le \xi_1, \\ (t-s)^3 - (1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i), \ s \le t, \xi_j \le s \le \xi_{j+1}, j = 1,2,...,m-3, \\ -(1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i), \ t \le s, \xi_j \le s \le \xi_{j+1}, j = 1,2,...,m-3, \\ (t-s)^3 - (1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s), \ \xi_{m-2} \le s \le t, \\ -(1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s), \ \xi_{m-2} \le s, t \le s. \end{cases}$$

Proof. Integrating the equation $u^{(4)}(t) = y(t)$ from 0 to t, we get

$$u'''(t) = \int_0^t y(\tau)d\tau + c.$$

As the similar process, combining the boundary conditions $u(1) = u'(0) = u''(0) = 0$, we have

$$\begin{aligned} u''(t) &= \int_0^t \int_0^\tau y(s)dsd\tau + ct \\ &= \int_0^t \int_s^t y(s)d\tau ds + ct \\ &= \int_0^t (t-s)y(s)ds + ct, \end{aligned}$$

$$\begin{aligned}
u'(t) &= \int_0^t \int_0^\tau (\tau - s)y(s)dsd\tau + \int_0^t c\tau d\tau \\
&= \int_0^t \int_s^t (\tau - s)y(s)d\tau ds + \frac{c}{2}t^2 \\
&= \frac{1}{2}\Big[\int_0^t (t-s)^2 y(s)ds + ct^2\Big]
\end{aligned}$$

and

$$\begin{aligned}
u(t) &= -\frac{1}{2}\Big[\int_t^1 \int_0^\tau (\tau - s)^2 y(s)dsd\tau + c\int_t^1 \tau^2 d\tau\Big] \\
&= -\frac{1}{2}\Big[\int_t^1 \int_0^t (\tau - s)^2 y(s)dsd\tau + \int_t^1 \int_t^\tau (\tau - s)^2 y(s)dsd\tau + \int_t^1 \tau^2 d\tau\Big] \\
&= -\frac{1}{2}\Big[\int_0^t \int_t^1 (\tau - s)^2 y(s)d\tau ds + \int_t^1 \int_s^1 (\tau - s)^2 y(s)d\tau ds + \frac{c}{3}(1-t^3)\Big] \\
&= -\frac{1}{6}\Big[\int_0^1 (1-s)^3 y(s)ds - \int_0^t (t-s)^3 y(s)ds + c(1-t^3)\Big].
\end{aligned}$$

Then by these above expressions, we have

$$u''(1) = \int_0^1 (1-s)y(s)ds + c$$

and

$$\sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = \sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} y(s)ds + c\sum_{i=1}^{m-2} \alpha_i.$$

Furthermore, from $u''(1) - \sum_{i=1}^{m-2} \alpha_i u'''(\xi_i) = 0$, it follows that

$$c = \frac{1}{1 - \sum_{i=1}^{m-2} \alpha_i}\Big[\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} y(s)ds - \int_0^1 (1-s)y(s)ds\Big],$$

which yields

$$u(t) = -\frac{1}{6}\Big[\int_0^1 (1-s)^3 y(s)ds - \int_0^t (t-s)^3 y(s)ds + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\sum_{i=1}^{m-2} \alpha_i \int_0^{\xi_i} y(s)ds - \int_0^1 (1-s)y(s)ds\Big)\Big].$$

If $s \le t$, $s \le \xi_1$, then

$$\begin{aligned}
G(t,s) &= -\frac{1}{6}[(1-s)^3 - (t-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(\sum_{i=1}^{m-2}\alpha_i - 1 + s)] \\
&= \frac{1}{6}[(t-s)^3 - (1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}].
\end{aligned}$$

If $t \le s \le \xi_1$, then

$$\begin{aligned}
G(t,s) &= -\frac{1}{6}[(1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(\sum_{i=1}^{m-2}\alpha_i - 1 + s)] \\
&= \frac{1}{6}[-(1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}].
\end{aligned}$$

If $s \leq t$, $\xi_j \leq s \leq \xi_{j+1}$ $(j = 1, 2..., m-3)$, then

$$\begin{aligned} G(t,s) &= -\frac{1}{6}[(1-s)^3 - (t-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(\sum_{i=j+1}^{m-2}\alpha_i - 1 + s)] \\ &= \frac{1}{6}[(t-s)^3 - (1-s)^3 + \frac{1-t^3}{1-\sum_{i=j+1}^{m-2}\alpha_i}(1 - s - \sum_{i=j+1}^{m-2}\alpha_i)]. \end{aligned}$$

If $t \leq s$, $\xi_j \leq s \leq \xi_{j+1}$ $(j = 1, 2..., m-3)$, then

$$\begin{aligned} G(t,s) &= -\frac{1}{6}[(1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(\sum_{i=j+1}^{m-2}\alpha_i - 1 + s)] \\ &= \frac{1}{6}[-(1-s)^3 + \frac{1-t^3}{1-\sum_{i=j+1}^{m-2}\alpha_i}(1 - s - \sum_{i=j+1}^{m-2}\alpha_i)]. \end{aligned}$$

If $s \leq t$, $\xi_{m-2} \leq s$, then

$$\begin{aligned} G(t,s) &= -\frac{1}{6}[(1-s)^3 - (t-s)^3 - \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)] \\ &= \frac{1}{6}[(t-s)^3 - (1-s)^3 + \frac{1-t^3}{1-\sum_{i=j+1}^{m-2}\alpha_i}(1-s)]. \end{aligned}$$

If $t \leq s$, $\xi_{m-2} \leq s$, then

$$\begin{aligned} G(t,s) &= -\frac{1}{6}[(1-s)^3 - \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)] \\ &= \frac{1}{6}[-(1-s)^3 + \frac{1-t^3}{1-\sum_{i=j+1}^{m-2}\alpha_i}(1-s)].\diamond \end{aligned}$$

Lemma 2.2. Assume that (H1) and (H2) hold. Then $G(t,s)$ satisfies the following properties.
(I) *Sign of* $G(t,s)$:

$$G(t,s) \geq 0 \quad \text{for } 0 \leq s < \xi_1, \; G(t,s) \leq 0 \quad \text{for } \xi_1 \leq s \leq 1.$$

(II) $Max - Min$ *value of* $G(t,s)$:

(1) for $0 \leq s < \xi_1$,

$$\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0,$$

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = \frac{1}{6}\Big[1 - (1-s)^3 - \frac{s}{1-\sum_{i=1}^{m-2}\alpha_i}\Big];$$

(2) for $\xi_1 \leq s \leq \xi_{m-2}$,

$$\min\{G(t,s) : t \in [0,1]\} = G(0,s) = -\frac{1}{6}\Big[(1-s)^3 + \frac{1-s}{1-\sum_{i=1}^{m-2}\alpha_i}\Big],$$

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0;$$

(3) for $\xi_{m-2} \le s \le 1$,

$$\min\{G(t,s) : t \in [0,1]\} = G(0,s) = -\frac{1}{6}\Big[(1-s)^3 + \frac{1-s-\sum_{i=j+1}^{m-2}\alpha_i}{1-\sum_{i=1}^{m-2}\alpha_i}\Big],$$

$$\max\{G(t,s) : t \in [0,1]\} = G(1,s) = 0.$$

Proof. For $0 \le s < \xi_1$, by the definition of $G(t,s)$, it is clear that $G(t,s)$ is continuous and derivativable with respect to t at $[0,1]$. On one hand, if $s \le t \le 1$, we have

$$\frac{\partial G}{\partial t} = \frac{1}{2}[(t-s)^2 - t^2 + \frac{st^2}{1-\sum_{i=1}^{m-2}\alpha_i}] \le 0.$$

On the other hand, if $0 \le t < s$, we have

$$\frac{\partial G}{\partial t} = \frac{1}{2}(-t^2 + \frac{st^2}{1-\sum_{i=1}^{m-2}\alpha_i}) \le 0,$$

So $G(t,s)$ is nonincreasing on t, which yields that

$$\min\{G(t,s) : t \in [0,1]\} = G(1,s) = 0,$$

$$\max\{G(t,s) : t \in [0,1]\} = G(0,s) = \frac{1}{6}[1-(1-s)^3 - \frac{s}{1-\sum_{i=1}^{m-2}\alpha_i}]$$

and

$$G(t,s) \ge 0 \quad \text{for } 0 \le s < \xi_1.$$

In the similar way, we can get $G(t,s) \le 0$ for $\xi_1 \le s < 1$. The corresponding maximum and minimum value of $G(t,s)$ also can be derived.$\diamond$

Lemma 2.3. Assume that (H1) and (H2) hold. Then for any given $y(t) \in K_0$, the solution $u(t)$ of LBVP (2.1) satisfies $u(t) \in K_0$, and $u(t)$ is concave on $[0,\xi_1]$.

proof. For $t \in [0,\xi_1]$, we have

$$\begin{aligned} u(t) &= \frac{1}{6}\Big\{\int_0^t \Big[(t-s)^3 - (1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]y(s)ds \\ &\quad + \int_t^{\xi_1}\Big[-(1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]y(s)ds - \int_{\xi_1}^1 (1-s)^3 y(s)ds \\ &\quad + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)y(s)ds + \int_{\xi_{m-2}}^1 (1-s)y(s)ds\Big)\Big\}. \end{aligned}$$

Since $y(t) \in K_0$ and $\frac{1}{2} \le \sqrt{\frac{5\sum_{i=1}^{m-2}\alpha_i - 4}{12\sum_{i=1}^{m-2}\alpha_i - 14}} < \xi_1 < \xi_2 < ... < \xi_{m-2} < 1$, via some computations, we have

$$\begin{aligned}
u'(t) &= \frac{1}{2}\Big[\int_0^t \Big(s(s-2t) + \frac{st^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)y(s)ds + \int_t^{\xi_1}(-t^2 + \frac{st^2}{1-\sum_{i=1}^{m-2}\alpha_i})y(s)ds \\
&\quad - \frac{t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)y(s)ds + \int_{\xi_{m-2}}^1(1-s)y(s)ds\Big)\Big] \\
&\le \frac{1}{2}y(\xi_1)\Big[\int_0^t \Big(s(s-2t) + \frac{st^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)ds + \int_t^{\xi_1}(-t^2 + \frac{st^2}{1-\sum_{i=1}^{m-2}\alpha_i})ds \\
&\quad - \frac{t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)ds + \int_{\xi_{m-2}}^1(1-s)ds\Big)\Big] \\
&= \frac{1}{6}y(\xi_1)\Big[t^2(t-3\xi_1) - \frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\frac{1}{2} - \xi_1 + \sum_{i=2}^{m-2}\alpha_i(\xi_1-\xi_i)\Big)\Big] \\
&\le 0
\end{aligned}$$

and

$$\begin{aligned}
u''(t) &= \int_0^t(-s + \frac{st}{1-\sum_{i=1}^{m-2}\alpha_i})y(s)ds + \int_t^{\xi_1}(-t + \frac{ts}{1-\sum_{i=1}^{m-2}\alpha_i})y(s)ds \\
&\quad - \frac{t}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)y(s)ds + \int_{\xi_{m-2}}^1(1-s)y(s)ds\Big] \\
&\le y(\xi_1)\Big[\int_0^t(-s + \frac{st}{1-\sum_{i=1}^{m-2}\alpha_i})ds + \int_t^{\xi_1}(-t + \frac{ts}{1-\sum_{i=1}^{m-2}\alpha_i})ds \\
&\quad - \frac{t}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)ds + \int_{\xi_{m-2}}^1(1-s)ds\Big)\Big] \\
&\le \frac{1}{2}y(\xi_1)\left[t(t-2\xi_1) - \frac{2t}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\frac{1}{2} - \xi_1 - \sum_{i=2}^{m-2}\alpha_i(\xi_i-\xi_1)\Big)\right] \\
&\le 0,
\end{aligned}$$

which imply that $u(t)$ is decreasing and concave on $[0,\xi_1]$.

For $t \in [\xi_k, \xi_{k+1}]$ $(k = 1, 2, ..., m-3)$ and $y \in K_0$, from the expression

$$\begin{aligned}
u(t) &= \frac{1}{6}\Big[\int_0^{\xi_1}\Big((t-s)^3 - (1-s)^3 + 1 - t^3 - \frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}\Big)y(s)ds \\
&\quad + \sum_{j=1}^{k-1}\int_{\xi_j}^{\xi_{j+1}}\Big((t-s)^3 - (1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)\Big)y(s)ds \\
&\quad + \int_{\xi_k}^t\Big((t-s)^3 - (1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=k+1}^{m-2}\alpha_i)\Big)y(s)ds \\
&\quad + \int_t^{\xi_{k+1}}\Big(-(1-s)^3 + \frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=k+1}^{m-2}\alpha_i)\Big)y(s)ds
\end{aligned}$$

$$+\sum_{j=k+1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}\Big(-(1-s)^3+\frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)\Big)y(s)ds$$
$$+\int_{\xi_{m-2}}^{1}\Big(-(1-s)^3+\frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)\Big)y(s)ds\Big],$$

it follows that

$$\begin{aligned}
u'(t) &= \frac{1}{6}\Bigg\{\int_0^{\xi_1}\Big[3(t-s)^2-3t^2+\frac{3st^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]y(s)ds \\
&\quad+\sum_{j=1}^{k-1}\int_{\xi_j}^{\xi_{j+1}}\Big[3(t-s)^2-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)\Big]y(s)ds \\
&\quad+\int_{\xi_k}^{t}\Big[3(t-s)^2-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=k+1}^{m-2}\alpha_i)\Big]y(s)ds \\
&\quad-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\int_t^{\xi_{k+1}}(1-s-\sum_{i=k+1}^{m-2}\alpha_i)y(s)ds \\
&\quad+\sum_{j=k+1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)y(s)ds+\int_{\xi_{m-2}}^{1}(1-s)y(s)ds\Big]\Bigg\} \\
&\le \frac{1}{6}y(\xi_1)\Big[3\Big(\int_0^t(t-s)^2ds-\int_0^{\xi_1}t^2ds\Big) \\
&\quad-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(-\int_0^{\xi_1}sds+\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)ds+\int_{\xi_{m-2}}^{1}(1-s)ds\Big)\Big] \\
&= \frac{1}{6}y(\xi_1)\Bigg[t^2(t-3\xi_1)-\frac{3t^3}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\frac{1}{2}-\xi_1+\sum_{i=2}^{m-2}\alpha_i(\xi_1-\xi_i)\Big)\Bigg].
\end{aligned}$$

Since $\frac{1}{2}\le\sqrt{\frac{5\sum_{i=1}^{m-2}\alpha_i-4}{12\sum_{i=1}^{m-2}\alpha_i-14}}\le\xi_1\le\xi_2\le\ldots\le\xi_{m-2}\le1$, it is clear that $u'(t)\le0$, which implies that $u(t)$ is decreasing.

For $t\in[\xi_{m-2},1]$ and $y\in K_0$, from the expression

$$\begin{aligned}
u(t) &= \frac{1}{6}\Bigg\{\int_0^{\xi_1}\Big[(t-s)^3-(1-s)^3+1-t^3-\frac{s(1-t^3)}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]y(s)ds \\
&\quad+\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}\Big[(t-s)^3-(1-s)^3+\frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)\Big]y(s)ds \\
&\quad+\int_{\xi_{m-2}}^{t}\Big[(t-s)^3-(1-s)^3+\frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)\Big]y(s)ds \\
&\quad+\int_t^1\Big[-(1-s)^3+\frac{1-t^3}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)\Big]y(s)ds\Bigg\},
\end{aligned}$$

it follows that

$$\begin{aligned}
u'(t) &= \frac{1}{6}\Bigg\{\int_0^{\xi_1}\Big[3(t-s)^2-3t^2+\frac{3st^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]y(s)ds \\
&\quad+\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}\Big[3(t-s)^2-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)\Big]y(s)ds \\
&\quad+\int_{\xi_{m-2}}^{t}\Big[3(t-s)^2-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)\Big]y(s)ds-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\int_t^1(1-s)y(s)ds\Bigg\} \\
&\le \frac{1}{6}y(\xi_1)\Bigg\{\int_0^{\xi_1}\Big[3(t-s)^2-3t^2+\frac{3st^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]ds \\
&\quad+\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}\Big[3(t-s)^2-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)\Big]ds \\
&\quad+\int_{\xi_{m-2}}^{t}\Big[3(t-s)^2-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}(1-s)\Big]ds-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\int_t^1(1-s)ds\Bigg\} \\
&= \frac{1}{6}y(\xi_1)\Bigg[t^2(t-3\xi_1)-\frac{3t^2}{1-\sum_{i=1}^{m-2}\alpha_i}\Big(\frac{1}{2}-\xi_1+\sum_{i=2}^{m-2}\alpha_i(\xi_1-\xi_i)\Big)\Bigg].
\end{aligned}$$

Since $\frac{1}{2}\le\sqrt{\frac{5\sum_{i=1}^{m-2}\alpha_i-4}{12\sum_{i=1}^{m-2}\alpha_i-14}}\le\xi_1\le\xi_2\le\ldots\le\xi_{m-2}\le1$, it is clear that $u'(t)\le0$.

Therefore, combining the above discussion with the boundary condition $u(1)=0$, it yields that $u(t)$ is nonnegative and decreasing on $[0,1]$, and concave on $[0,\xi_1]$. $\diamond$

Lemma 2.4. Let $y(t)\in K_0$. Then the unique solution $u(t)$ of the LBVP (2.1) satisfies

$$\min_{t\in[0,\tau]}u(t)\ge\tau^*\|u\|,$$

where $\tau\in(0,\frac{1}{2}]$ and $\tau^*=\frac{\xi_1-\tau}{\xi_1}$.

proof. From Lemma 2.3, we know that $u(t)$ is concave on $[0,\xi_1]$. Thus for $t\in[0,\xi_1]$, we have

$$u(t)\ge\frac{\xi_1-t}{\xi_1}u(0)+\frac{t}{\xi_1}u(\xi_1).$$

Since $u(t)\in K_0$ and $\|u\|=u(0)$, it yields

$$u(t)\ge\frac{\xi_1-t}{\xi_1}\|u\|.$$

Therefore, via some computations, we have

$$\min_{t\in[0,\tau]}u(t)\ge\tau^*\|u\|.\diamond$$

Lemma 2.5. Assume that (H1) and (H2) hold. Then there exists a $\theta\in(0,\xi_1)$ such that

$$\int_\theta^{\xi_1}G(0,s)ds+\int_{\xi_1}^1G(0,s)ds\ge0.$$

proof. Let

$$g(x)=\xi_1-x-\frac{1}{4}(1-x)^4+\frac{x^2}{2(1-\sum_{i=1}^{m-2}\alpha_i)}+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\frac{1}{2}-\xi_1-\sum_{i=2}^{m-2}\alpha_i(\xi_i-\xi_1)\Big],\ x\in[0,\xi_1].$$

Then $g'(x)=(1-x)^3-1+\frac{x}{1-\sum_{i=1}^{m-2}\alpha_i}\le 0$, which implies that $g(x)$ is decreasing.

Since

$$g(0)=\xi_1-\frac{1}{4}+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\frac{1}{2}-\xi_1-\sum_{i=2}^{m-2}\alpha_i(\xi_i-\xi_1)\Big]>0$$

and

$$\begin{aligned}
g(\xi_1) &= \xi_1-\xi_1-\frac{1}{4}(1-\xi_1)^4+\frac{\xi_1^2}{2(1-\sum_{i=1}^{m-2}\alpha_i)}+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\frac{1}{2}-\xi_1-\sum_{i=2}^{m-2}\alpha_i(\xi_i-\xi_1)\Big]\\
&= -\frac{1}{4}(1-\xi_1)^4+\frac{\xi_1^2}{2(1-\sum_{i=1}^{m-2}\alpha_i)}+\frac{1}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\frac{1}{2}-\xi_1-\sum_{i=2}^{m-2}\alpha_i(\xi_i-\xi_1)\Big]\\
&= -\frac{1}{4}(1-\xi_1)^4+\frac{1}{2(1-\sum_{i=1}^{m-2}\alpha_i)}\Big[(1-\xi_1)^2-2\sum_{i=2}^{m-2}\alpha_i(\xi_i-\xi_1)\Big]\\
&\le -\frac{1}{4}(1-\xi_1)^4+\frac{1}{2(1-\sum_{i=1}^{m-2}\alpha_i)}\Big[(1-\xi_1)^2-2\sum_{i=2}^{m-2}\alpha_i(1-\xi_1)\Big]\\
&= -\frac{1}{4}(1-\xi_1)^4+\frac{1}{2(1-\sum_{i=1}^{m-2}\alpha_i)}(1-\xi_1)(1-\xi_1-2\sum_{i=2}^{m-2}\alpha_i)\\
&< 0,
\end{aligned}$$

by the continuity of g, it yields that there exists a unique $x_0\in(0,\xi_1)$ such that $g(x_0)=0$. Choosing $\theta\in(0,x_0]$, then

$$\int_\theta^{\xi_1}G(0,s)ds+\int_{\xi_1}^1 G(0,s)ds=\frac{1}{6}g(\theta)\ge\frac{1}{6}g(x_0)=0.\diamond$$

Now let $K=\{u\in K_0:\ \min_{t\in[0,\tau]}u(t)\ge\tau^*\|u\|\}$ and define an operator T by

$$(Tu)(t)=\int_0^1 G(t,s)f(s,u(s))ds.$$

It is obvious that if u is a fixed point of T in K, then u is a nonnegative and decreasing solution of BVP (1.1). Finally, by Lemmas 2.3 and 2.4 and the standard argument, we can get the following conclusion.

Lemma 2.6. Assume that (H1),(H2) and (H3) hold. Then $T:K\to K$ and T is completely continuous operator.

3. **Main Results**

For convenience, we denote

$$\theta^*=1-\frac{\theta}{\xi_1},$$

$$A = \int_0^{\tau} G(\xi_1, s)ds, \ B = \int_0^{\xi_1} G(\tau, s)ds,$$

$$P = \int_0^{\xi_1} G(0, s)ds \text{ and } Q = \int_0^{\theta} G(0, s)ds.$$

Obviously, $0 < \theta^* < 1$ and $0 < Q < P$.

Theorem 3.1. Assume that (H1), (H2) and (H3) hold. If there exist two positive constants r_1, r_2 with $r_1 < r_2$ such that

$$f(0, r_1) < \frac{r_1}{P}, \ f(\theta, \theta^* r_2) > \frac{r_2}{Q},$$

then BVP (1.1) has at least one decreasing positive solution $u(t)$ satisfying

$$r_1 < \|u\| < r_2.$$

Proof. Let

$$\Omega_{r_i} = \{u \in K : \|u\| < r_i\}, i = 1, 2.$$

On one hand, for any $u \in \partial\Omega_{r_1}$, we have

$$0 \le u(t) \le \|u\| = r_1, \ t \in [0, 1].$$

Then by Lemma 2.2, Lemma 2.4 and (H3), we have

$$\begin{aligned}
\|Tu\| &= (Tu)(0) = \int_0^1 G(0, s)f(s, u(s))ds \\
&= \int_0^{\xi_1} G(0, s)f(s, u(s))ds + \int_{\xi_1}^1 G(0, s)f(s, u(s))ds \\
&\le \int_0^{\xi_1} G(0, s)f(s, u(s))ds \\
&\le f(0, r_1)\int_0^{\xi_1} G(0, s)ds \\
&< \frac{r_1}{P}\int_0^{\xi_1} G(0, s)ds \\
&= r_1 = \|u\|
\end{aligned}$$

namely, $\|Tu\| \le \|u\|$ for any $u \in \partial\Omega_{r_1}$. Hence, by (2) of Lemma 1.3, we have

$$i_K(T, \Omega_{r_1}) = 1. \tag{3.1}$$

On the other hand, for any $u \in \partial\Omega_{r_2}$, we have

$$u(\theta) = \min_{s\in[0,\theta]} u(s) \ge \theta^*\|u\| = \theta^* r_2. \tag{3.2}$$

Let $e(t) \equiv 1$ for $t \in [0, 1]$. Then it is obvious that $e \in K\backslash\{0\}$. Now, we prove that $u \ne Tu + \lambda e$ for all $u \in \partial\Omega_{r_2}$ and all $\lambda \ge 0$. Suppose on the contrary that there exist $u^* \in \partial\Omega_{r_2}$ and $\lambda^* \ge 0$ such that $u^* = Tu^* + \lambda^* e$. Then by Lemma 2.2, Lemma 2.4, Lemma 2.5, (H3) and(3.2), we have

$$\begin{aligned}
r_2 &= \|u^*\| = u^*(0) = (Tu^*)(0) + \lambda^* \\
&= \int_0^1 G(0,s)f(s,u^*(s))ds + \lambda^* \\
&= \int_0^\theta G(0,s)f(s,u^*(s))ds + \int_\theta^{\xi_1} G(0,s)f(s,u^*(s))ds + \int_{\xi_1}^1 G(0,s)f(s,u^*(s))ds + \lambda^* \\
&\geq f(\theta,u^*(\theta))\int_0^\theta G(0,s)ds + f(\xi_1,u^*(\xi_1))\left(\int_\theta^{\xi_1} G(0,s)ds + \int_{\xi_1}^1 G(0,s)ds\right) + \lambda^* \\
&\geq f(\theta,u^*(\theta))\int_0^\theta G(0,s)ds + \lambda^* \\
&\geq f(\theta,\theta^* r_2)\int_0^\theta G(0,s)ds + \lambda^* \\
&> \frac{r_2}{Q}\int_0^\theta G(0,s)ds + \lambda^* \\
&= r_2 + \lambda^*,
\end{aligned}$$

which mens a contradiction. So it concludes that $u \neq Tu + \lambda e$ for all $u \in \partial\Omega_{r_2}$ and all $\lambda \geq 0$. Hence, by (1) of Lemma 1.3, it yields

$$i_K(T, \Omega_{r_2}) = 0. \tag{3.3}$$

Therefore, from (3.1), (3.3) and (3) of Lemma 1.3, it follows that T has a fixed point u in K, which is a solution of BVP (1.1).$\diamond$

As the similar proof of Theorem 3.1, we also can get

Theorem 3.2. Assume that (H1), (H2) and (H3) hold. If there exist two positive constants r_1, r_2 with $r_1 < r_2$ such that

$$f(\theta, \theta^* r_1) > \frac{r_1}{Q},\ f(0, r_2) < \frac{r_2}{P},$$

then BVP (1.1) has at least one decreasing positive solution $u(t)$ satisfying

$$r_1 < \|u\| < r_2.$$

Theorem 3.3. Assume that (H1), (H2) and (H3) hold. In addition, f satisfies the following assumption
($H4$) there exist three constants a_1, a_2 and a_3 with $0 < a_1 < a_2 < \tau^* a_3$ such that

$$f(\tau, \tau^* a_1) > \frac{a_1}{A},\ f(0, \frac{a_2}{\tau^*}) < \frac{a_2}{B} \text{ and } f(\tau, a_3) > \frac{a_3}{A}.$$

Then BVP (1.1) has at least two positive and decreasing solutions.

proof. Define the increasing, nonnegative, and continuous functionals γ, ω and ψ on K as follows:

$$\gamma(u) = \min_{t\in[0,\tau]} u(t) = u(\tau),$$

$$\omega(u)=\max_{t\in[\tau,1]}u(t)=u(\tau)$$

and

$$\psi(u)=\max_{t\in[0,1]}u(t)=u(0).$$

Obviously, for any $u\in K$, we have

$$\gamma(u)=\omega(u)\le\psi(u)\text{ and }\|u\|\le\frac{1}{\tau^*}\gamma(u).$$

Furthermore, it is clear that

$$\omega(\lambda u)=\lambda\omega(u)\text{ for }0\le\lambda\le1,\ u\in K.$$

Now, for any $u\in K$, we claim that

$$\int_\tau^1 G(\xi_1,s)f(s,u(s))ds\ge0. \tag{3.4}$$

In fact, from (H1),(H2),(H3) and $\tau\in(0,\frac12]$, we have

$$\begin{aligned}
&\int_\tau^1 G(\xi_1,s)f(s,u(s))ds\\
=\ &\int_\tau^{\xi_1}G(\xi_1,s)f(s,u(s))ds+\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}G(\xi_1,s)f(s,u(s))ds+\int_{\xi_{m-2}}^1 G(\xi_1,s)f(s,u(s))ds\\
\ge\ &\frac16 f(\xi_1,u(\xi_1))\Big\{\int_\tau^{\xi_1}\Big[(\xi_1-s)^3+1-\xi_1^3-\frac{s(1-\xi_1^3)}{1-\sum_{i=1}^{m-2}\alpha_i}\Big]ds-\int_\tau^1(1-s)^3ds\\
&+\frac{1-\xi_1^3}{1-\sum_{i=1}^{m-2}\alpha_i}\Big[\sum_{j=1}^{m-3}\int_{\xi_j}^{\xi_{j+1}}(1-s-\sum_{i=j+1}^{m-2}\alpha_i)ds+\int_{\xi_{m-2}}^1(1-s)ds\Big]\Big\}\\
=\ &\frac16 f(\xi_1,u(\xi_1))\Big\{\frac14\Big[(\xi_1-\tau)^4-(1-\tau)^4\Big]+(1-\xi_1^3)(\xi_1-\tau)+\frac{(1-\xi_1^3)\tau^2}{2(1-\sum_{i=1}^{m-2}\alpha_i)}\\
\ge\ &\frac{1-\xi_1}{24}f(\xi_1,u(\xi_1))\Big[3\xi_1^3+3\xi_1^2+3\xi_1-1+4\tau^3+\Big(\frac{2(\xi_1^2+\xi_1+1)}{1-\sum_{i=1}^{m-2}\alpha_i}-6\xi_1-6\Big)\tau^2\Big]\\
\ge\ &\frac{1-\xi_1}{24}f(\xi_1,u(\xi_1))\Big[3\xi_1^3+3\xi_1^2+3\xi_1-1+\Big(\frac{2(\xi_1^2+\xi_1+1)}{1-\sum_{i=1}^{m-2}\alpha_i}-6\xi_1-6\Big)\tau^2\Big]\\
\ge\ &\frac{1-\xi_1}{24}f(\xi_1,u(\xi_1))\Big[3\xi_1^3+3\xi_1^2+3\xi_1-1-\frac{3\xi_1}{2}-\frac32-\frac{\xi_1^2+\xi_1+1}{2(\sum_{i=1}^{m-2}\alpha_i-1)}\Big]\\
=\ &\frac{1-\xi_1}{24}f(\xi_1,u(\xi_1))\Big[3\xi_1^3+\frac{6\sum_{i=1}^{m-2}\alpha_i-7}{2(\sum_{i=1}^{m-2}\alpha_i-1)}\xi_1^2+\frac{3\sum_{i=1}^{m-2}\alpha_i-4}{2(\sum_{i=1}^{m-2}\alpha_i-1)}\xi_1-\frac{5\sum_{i=1}^{m-2}\alpha_i-4}{2(\sum_{i=1}^{m-2}\alpha_i-1)}\Big]\\
\ge\ &\frac{1-\xi_1}{48(\sum_{i=1}^{m-2}\alpha_i-1)}f(\xi_1,u(\xi_1))\Big[(12\sum_{i=1}^{m-2}\alpha_i-14)\xi_1^2-(5\sum_{i=1}^{m-2}\alpha_i-4)\Big]\\
\ge\ &0.
\end{aligned}$$

First, for any $u\in\partial K(\gamma,a_3)$, it means that $u\in K$ and $\gamma(u)=u(\tau)=a_3$. Then

$$u(t)\ge u(\tau)=a_3,t\in[0,\tau]. \tag{3.5}$$

Since $(Tu)(t)$ is decreasing on $[0,1]$, from (3.5), (H3) and (H4) it follows

$$\begin{aligned}\gamma(Tu) &= (Tu)(\tau) \geq (Tu)(\xi_1) = \int_0^1 G(\xi_1,s)f(s,u(s))ds \\ &\geq \int_0^\tau G(\xi_1,s)f(s,u(s))ds \\ &\geq \int_0^\tau G(\xi_1,s)f(\tau,a_3)ds \\ &> \frac{a_3}{A}\int_0^\tau G(\xi_1,s)ds = a_3,\end{aligned}$$

which means that $\gamma(Tu) > a_3$ for all $u \in \partial K(\gamma, a_3)$.

Second, for any $u \in \partial K(\omega, a_2))$, namely, $u \in K$ and $\omega(u) = a_2$. Since $\|u\| \leq \frac{1}{\tau^*}\gamma(u) = \frac{1}{\tau^*}\omega(u)$, we have

$$0 \leq u(t) \leq \|u\| \leq \frac{a_2}{\tau^*}, t \in [0,\xi_1]. \tag{3.6}$$

Then by Lemma 2.2, (3.6), (H3) and (H4), we get

$$\begin{aligned}\omega(Tu) &= (Tu)(\tau) = \int_0^1 G(\tau,s)f(s,u(s))ds \\ &\leq \int_0^{\xi_1} G(\tau,s)f(s,u(s))ds \\ &\leq \int_0^{\xi_1} G(\tau,s)f(0,\frac{a_2}{\tau^*})ds \\ &< \frac{a_2}{B}\int_0^{\xi_1} G(\tau,s)ds = a_2,\end{aligned}$$

which means that $\omega(Tu) < a_2$ for all $u \in \partial K(\omega, a_2)$.

Finally, it is clear that $\frac{a_1}{2} \in K(\psi, a_1)$, then $K(\psi, a_1)$ is nonempty. Moreover, for $u \in \partial K(\psi, a_1)$, namely, $u \in K$ and $\psi(u) = u(0) = a_1$. Then

$$\begin{aligned}\psi(Tu) &= (Tu)(0) \\ &\geq (Tu)(\xi_1) \\ &= \int_0^1 G(\xi_1,s)f(s,u(s))ds \\ &\geq \int_0^\tau G(\xi_1,s)f(\tau,\tau^*a_1)ds \\ &> \frac{a_1}{A}\int_0^\tau G(\xi_1,s)ds = a_1,\end{aligned}$$

which means that $\psi(Tu) > a_1$ for all $u \in \partial K(\psi, a_1)$.

So, all the hypotheses of Lemma 1.4 are satisfied. Hence, T has at least two fixed points u_1 and u_2 satisfying

$$a_1 < \max_{t\in[0,1]} u_1(t) \text{ with } \max_{t\in[\tau,1]} u_1(t) < a_2,$$

$$a_2 < \max_{t\in[\tau,1]} u_2(t) \text{ with } \min_{t\in[0,\tau]} u_2(t) < a_3.\diamond$$

Theorem 3.4. Assume that (H1), (H2) and (H3) hold. Moreover, suppose that

there exist positive real numbers a, b, d with $0 < a < b < \tau^* d$ such that
$(H5)$ $f(t,u) \le \frac{d}{P}$, $(t,u) \in [0,1] \times [0,d]$;
$(H6)$ $f(t,u) > \frac{b}{B}$, $(t,u) \in [0,\tau] \times [b, \frac{b}{\tau^*}]$;
$(H7)$ $f(t,u) < \frac{a}{P}$, $(t,u) \in [0,1] \times [0,a]$.
Then BVP (1.1) has at least three positive solutions.

proof. Now we shall show that T satisfies the conditions of the Avery-Peterson fixed point theorem. Let

$$\psi(u) = \varphi(u) = \phi(u) = \max_{t\in[0,1]} u(t) = u(0)$$

and

$$\alpha(u) = \min_{t\in[0,\tau]} u(t) = u(\tau).$$

It is obvious that ψ, ϕ and φ are nonnegative, continuous convex functionals on K, α is a nonnegative, continuous concave functional on K.

First, we show that $T : K(\phi, d) \to K(\phi, d)$. If $u \in K(\phi, d)$, then

$$\phi(u) = \max_{t\in[0,1]} u(t) \le d.$$

By Lemma 2.2, we can get

$$\begin{aligned}
\phi(Tu) &= (Tu)(0) \\
&= \int_0^1 G(0,s) f(s, u(s)) ds \\
&\le \int_0^{\xi_1} G(0,s) f(s, u(s)) ds \\
&\le \frac{d}{P} \int_0^{\xi_1} G(0,s) ds \\
&= \frac{d}{P} \cdot P = d
\end{aligned}$$

which yields that $T : K(\phi, d) \to K(\phi, d)$.

Choosing $\tilde{u}(t) \equiv \frac{b}{\tau^*} (0 \le t \le 1)$, then we have

$$\phi(\tilde{u}) = \frac{b}{\tau^*} \le d,\ \varphi(\tilde{u}) = \frac{b}{\tau^*},\ \alpha(\tilde{u}) = \frac{b}{\tau^*} > b,$$

which implies that $\tilde{u}(t) \in \{u \in K(\phi, \varphi, \alpha; b, \frac{b}{\tau^*}, d) : \alpha(u) > b\}$. Namely,

$$\{u \in K(\phi, \varphi, \alpha; b, \frac{b}{\tau^*}, d) : \alpha(u) > b\} \ne \emptyset.$$

If $u \in K(\phi, \varphi, \alpha; b, \frac{b}{\tau^*}, d)$, then $b \le u(t) \le \frac{b}{\tau^*}$, $0 \le t \le \tau$. Furthermore, we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[0,\tau]}(Tu)(t) = Tu(\tau) \\
&\ge Tu(\xi_1) \\
&= \int_0^1 G(\xi_1, s) f(s, u(s))ds \\
&\ge \int_0^\tau G(\xi_1, s) f(s, u(s))ds \\
&> \frac{b}{B}\int_0^\tau G(\xi_1, s)ds \\
&= \frac{b}{B}\cdot B = b,
\end{aligned}$$

which implies that (i) of Lemma 1.5 holds.

Third, if $u \in K(\phi, \alpha; b, d)$ and $\varphi(Tu) > c = \frac{b}{\tau^*}$, then by Lemma 2.4 and $Tu \in K$, we have

$$\begin{aligned}
\alpha(Tu) &= \min_{t\in[0,\tau]}(Tu)(t) \\
&\ge \tau^*\|Tu\| \\
&= \tau^*\varphi(Tu) \\
&> \tau^*\cdot\frac{b}{\tau^*} = b,
\end{aligned}$$

which implies that (ii) of Lemma 1.5 is satisfied.

Finally, if $u \in K(\phi, \psi; a, d)$ and $\psi(u) = \max_{t\in[0,1]} u(t) \ge a > 0$, then $\psi(0) = 0$ and $0 \notin K(\phi, \psi; a, d)$. By Lemma 2.2, we have

$$\begin{aligned}
\psi(Tu) &= (Tu)(0) \\
&= \int_0^1 G(0, s) f(s, u(s))ds \\
&\le \int_0^{\xi_1} G(0, s) f(s, u(s))ds \\
&< \frac{a}{P}\int_0^{\xi_1} G(0, s)ds \\
&= \frac{a}{P}\cdot P = a,
\end{aligned}$$

which implies that (iii) of Lemma 1.5 is satisfied.

Therefore, by Lemma 1.5, T has at least three fixed points u_1, u_2 and u_3; that is, BVP (1.1) has at least three positive and decreasing solutions u_1, u_2 and u_3 satisfying

$$\max_{t\in[0,1]} u_i(t) \le d \; for \; i = 1, 2, 3;$$

$$b < \min_{t\in[0,\tau]} u_1(t);$$

$$a < \max_{t\in[0,1]} u_2(t) \text{ with } \min_{t\in[0,\tau]} u_2(t) < b;$$

and

$$\max_{t\in[0,1]} u_3(t) < a.\diamond$$

4. **Examples**

Now we give two examples to illustrate Theorem 3.3 and 3.4.

Example 4.1. Consider the following BVP

$$(4.1) \qquad \begin{cases} u^{(4)}(t) = 1 - t + 2u^2(t), t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, u''(1) - 2u'''(\frac{4}{5}) - \frac{1}{8}u'''(\frac{6}{7}) = 0. \end{cases}$$

It is easy to verify that (H1)-(H3) hold.

If we choose $\tau = \frac{1}{3}$, then $\tau^* = \frac{7}{12}$. Moreover, via some calculations, we obtain

$$A = \int_0^{\frac{1}{3}} G(\frac{4}{5}, s)ds = \frac{1553}{121500}, \ B = \int_0^{\frac{4}{5}} G(\frac{1}{3}, s)ds = \frac{484643}{3645000}.$$

Furthermore, choosing $a_1 = 0.008$, $a_2 = 0.16$ and $a_3 = 40$. we also can verify that

$$f(\tau, a_3) = f(\frac{1}{3}, a_3) > \frac{121500}{1553}a_3,$$

$$f(0, \frac{a_2}{\tau^*}) = f(0, \frac{12a_2}{7}) < \frac{3645000}{484643}a_2$$

and

$$f(\tau, \tau^* a_1) = f(\frac{1}{3}, \frac{7a_1}{12}) > \frac{121500}{1553}a_1,$$

which imply that (H4) holds.

Therefore, by Theorem 3.3, BVP (4.1) has at least two positive and decreasing solutions.$\diamond$

Example 4.2. Consider the following BVP

$$(4.2) \qquad \begin{cases} u^{(4)}(t) = f(t,u), t \in [0,1], \\ u'(0) = u''(0) = u(1) = 0, u''(1) - 2u'''(\frac{4}{5}) - \frac{1}{8}u'''(\frac{6}{7}) = 0, \end{cases}$$

where

$$f(t,u) = \begin{cases} 25\sqrt{u} - \frac{27}{100}t^3 + \frac{997}{50}, \ (t,u) \in [0,1] \times [0,8], \\ u^2 - \frac{27}{100}t^3 + \frac{64}{25}, \ (t,u) \in [0,1] \times [8, +\infty). \end{cases}$$

It is easy to verify that (H1)-(H3) hold. We also take $\tau = \frac{1}{3}$ and $P = \int_0^{\frac{4}{5}} G(0,s)ds = \frac{59}{1994}$. Now choosing $a = 4$, $b = 8$ and $d = 30$. Then via some computations, we can verify that

$$f(t,u) \le \frac{1994}{59}d, \ \text{for } (t,u) \in [0,1] \times [0,d],$$

$$f(t,u) > \frac{3645000}{484643}b, \ \text{for } (t,u) \in [0,\frac{1}{3}] \times [b, \frac{12}{7}b]$$

and

$$f(t,u) < \frac{1994}{59}a, \ \text{for } (t,u) \in [0,1] \times [0,a],$$

which imply that (H5)-(H7) hold.

Therefore, by Theorem 3.4, BVP (4.2) has at least three positive and decreasing solutions.$\diamond$

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