Chapter 1
Two Operator Boundary - Domain Integral Equations for variable coefficient Mixed BVP in 2D

Ayele Tsegaye G., Bekele Solomon T.

Abstract The formulation and analysis of two operator Boundary-Domain Integral Equation systems for variable coefficient mixed BVP with in two dimensional domain is discussed. To analyse the two-operator approach, we applied one of its linear versions to the mixed (Dirichlet-Neumann) BVP for a linear second-order scalar elliptic variable-coefficient PDE and reduced it to four different BDIE systems. The two-operator BDIE systems are equivalent to the mixed BVP and thus are uniquely solvable, while the corresponding boundary domain integral operators are invertible in the appropriate Sobolev-Slobodetski (Bessel potential) spaces.
Key words: Two-operator Boundary-Domain Integral Equations,Sobolev-Slobodetski spaces, equivalence, invertibility.

1.1 Introduction

Any phenomena in physical science and engineering can be modelled using mathematics. In some situations, this modeling lead to linear or non-linear differential, or integral equation or even integro-differential equations. These equations play a considerable role in understanding practical problems. Particularly, Partial Differential Equations (PDEs) with variable coefficients often arise in mathematical modeling of inhomogeneous media in solid mechanics, electro-magnetics, thermo conductivity, fluid flow through porous media, and other areas of physics and engineering.

Through centuries, different methods of solving such problems have been developed. In any method, it is crucial to investigate the existence and uniqueness of

Tsegaye Gedif Ayele
Addis Ababa University, Addis Ababa, Ethiopia,
e-mail: tsegayeg@math.aau.edu.et

Solomon Tesfaye Bekele
Addis Ababa University, Addis Ababa, Ethiopia,
e-mail: so_te2004@yahoo.com

solution.

However, explicit fundamental solutions are not available if the coefficients of the original PDE are not constant. Nevertheless, for a rather wide class of variable-coefficient PDEs it is possible to use instead an explicit parametrix (Levi function) associated with the fundamental solution of the corresponding frozen-coefficient PDEs, and reduce BVPs for such PDEs to systems of Boundary-Domain Integral Equations for further numerical solution of the latter, see e.g. [CMN09a], [CMN11], [Mik02], [Mik05b],[Mik06] and references therein. In this paper, the direct segregated two-operator BDIEs for the mixed BVP for a second order elliptic PDE with variable coefficient in 2D is considered.

However, this (one-operator) approach does not work when the fundamental solution of the frozen-coefficient PDE is not known explicitly (as e.g. in the Lamè system of anisotropic elasticity). To overcome this difficulty, one can apply the two-operator approach, formulated in [Mik05a] for a certain non-linear problem, that employs a parametrix of another (second) PDE, not related with the PDE in question, for reducing the BVP to a BDIE system. Since the second PDE is rather arbitrary, one can always chose it in such a way, that its parametrix is known explicitly. The simplest choice for the second PDE is the one with an explicit fundamental solution.

Let Ω be a domain in $\mathbb{R}^2$ bounded by simple closed infinitely differentiable curve $\partial\Omega$, the set of all infinitely differentiable function on Ω with compact support is denoted by $\mathscr{D}(\Omega)$. The function space $\mathscr{D}'(\Omega)$ consists of all continuous linear functionals over $\mathscr{D}(\Omega)$. For $s \in \mathbb{R}$, we denote by $H^s(\mathbb{R}^2)$ the Bessel potential space. Note that the space $H^1(\mathbb{R}^2)$ coincides with the Sobolev space $W_2^1(\mathbb{R}^2)$ with equivalent norm and $H^{-s}(\mathbb{R}^2)$ is the dual space to $H^s(\mathbb{R}^2)$. For any non-empty open set $\Omega \in \mathbb{R}^n$ we define $H^s(\Omega) = \{u \in \mathscr{D}'(\Omega) : u = U|_\Omega \quad \text{for some} \quad \mathrm{U} \in \mathbb{R}^{\mathrm{n}}\}$. The space $\tilde{H}^s(\Omega)$ is defined to be the closure of $\mathscr{D}(\Omega)$ with respect to the norm of $H^s(\mathbb{R}^n)$. For $s \in (-\frac{1}{2}, \frac{1}{2})$,$H^s(\Omega)$ can be identified with $\tilde{H}^s(\Omega)$, (see, e.g., [McL00], [HCW08]).

Formulation of Mixed BVP

Consider the following elliptic differential equation with scalar variable coefficient

$$Au(x) = A(x, \partial_x)u(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[a(x)\frac{\partial u(x)}{\partial x_i}\right] = f(x), \quad x \in \Omega, \tag{1.1}$$

where u is an unknown function and f is a given function in Ω. Also we assume that $a \in C^\infty(\mathbb{R}^2)$, $0 < a_{min} \leq a(x) \leq a_{max} < \infty, \quad \forall x \in \mathbb{R}^2$.

For a linear operator A, we introduce the following subspace of $H^1(\Omega)$, [Gri85], [Cos88]

$$H^{1,0}(\Omega;A) = \{g \in H^1(\Omega) : Ag \in L_2(\Omega)\}$$

endowed with the norm

$$\|g\|^2_{H^{1,0}(\Omega;A)} := \|g\|^2_{H^1(\Omega)} + \|Ag\|^2_{L_2(\Omega)}.$$

From the well known theorem of Gauss Ostrogradski, if $h \in C^1_0(\overline{\Omega})$, then

$$\int_\Omega \frac{\partial}{\partial x_i} h(x) dx = \int_{\partial\Omega} \gamma^+ h(x) n_i(x) dS_x, \quad i = 1,2, \tag{1.2}$$

where $\gamma^+ h(x) = \lim_{\Omega \ni y \to x \in \partial\Omega} h(y)$ for $x \in \Omega$ is the interior boundary trace of $h(x)$. By the trace theorem (see, e.g., [McL00, Theorem 3.29,Theorem 3.38]), the integral relation (1.2) holds for any $h \in H^1(\Omega)$. Now for $u \in H^2(\Omega)$ and $v \in H^1(\Omega)$ if we put $h(x) = a(x)\frac{\partial u(x)}{\partial x_i} v(x)$ and apply the Gauss-Ostrogradski theorem, we obtain the following Green's first formula

$$\mathscr{E}_a(u,v) = -\int_\Omega (Au)(x)v(x)dx + \int_{\partial\Omega} T_a^+ u(x)\gamma^+ v(x) dS_x, \tag{1.3}$$

where $\mathscr{E}_a(u,v) = \sum_{i=1}^2 \int_\Omega a(x) \frac{\partial u(x)}{\partial x_j} \frac{\partial v(x)}{\partial x_j}$ is the symmetric bilinear form and

$$T_a^+ u(x) = \lim_{\Omega \ni y \to x \in \partial\Omega} \left[\sum_{i=1}^2 n_i(x) a(y) \frac{\partial}{\partial y_i} u(y) \right] \quad \text{for } x \in \partial\Omega \tag{1.4}$$

where $n(x)$ is the exterior (to Ω)unit normal at the point $x \in \Omega$, is the interior co-normal derivative.

For $u \in \mathrm{H}^1(\Omega)$ the classical co-normal differentiation operators on $\partial\Omega$ do not generally exist in the trace sense (1.4). However, if $u \in \mathrm{H}^{1,0}(\Omega;A)$, one can define the generalized (canonical) co-normal derivative $T_a^+ u \in \mathrm{H}^{-\frac{1}{2}}(\partial\Omega)$ with the help of the first Green identity (1.3) (see, e.g., [Cos88] [Mik11])

$$\left\langle T_a^+ u, w \right\rangle_{\partial\Omega} := \int_\Omega [(\gamma_{-1}^+ w) Au + E_a(u, \gamma_{-1}^+ w)] dx \qquad \forall w \in H^{\frac{1}{2}}(\partial\Omega), \tag{1.5}$$

where

$$E_a(u,v) := \sum_{i=1}^2 a(x) \frac{\partial u(x)}{\partial x_i} \frac{\partial v(x)}{\partial x_i}$$

and $\gamma_{-1}^+ : H^{\frac{1}{2}}(\partial\Omega) \to H^1(\Omega)$ is a continuous right inverse of the continuous interior trace operator $\gamma^+ : H^1(\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ while $\langle .,. \rangle$ denotes the duality brackets between the spaces $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$, extending the usual L_2 scalar product while
$E_a(u,v) = a(x)\nabla u(x).\nabla v(x)$ is the symmetric bilinear form.

For $u \in H^{1,0}(\Omega,\Delta), v \in H^1(\Omega)$ the first Green identity holds, [Cos88, Lemma 3.4],

$$\int_\Omega v(x)Au(x)dx = \int_{\partial\Omega} \gamma^+ v(x) T_a^+ u(x) dS(x) - \int_\Omega E_a(u,v)dx. \tag{1.6}$$

If $u \in H^2(\Omega)$, the canonical co-normal derivative $T_a^+ u$ defined by (1.5) reduces to its classical form (1.4).

We will derive and investigate the two operator boundary-domain integral equation system for the following mixed boundary value problem. Find a function $u \in H^1(\Omega)$ satisfying,

$$Au = f \qquad \text{in} \quad \Omega, \tag{1.7}$$

$$\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial_D\Omega, \tag{1.8}$$

$$T_a^+ u = \psi_0 \qquad \text{on} \quad \partial_N\Omega, \tag{1.9}$$

where $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega),\ \psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ and $f \in L_2(\Omega)$. Equation (1.7) is understood in the distributional sense and condition (1.8) in the trace sense while equality (1.9) is understood in the functional sense.

Remark 1. Here $H^{\frac{1}{2}}(\partial_D\Omega) = \{r_{\partial_D\Omega} g : g \in H^{\frac{1}{2}}(\partial\Omega)\}$ where $r_{\partial_D\Omega}$ is the restriction operator on $\partial_D\Omega$ and $H^{-\frac{1}{2}}(\partial_N\Omega)$ is the dual space of the subspace $\tilde{H}^{\frac{1}{2}}(\partial_N\Omega) = \{g : g \in H^{\frac{1}{2}}(\partial\Omega), supp(g) \subset \overline{\partial_N\Omega}\}$

Here the BVP defined (1.7) - (1.9) has important applications in engineering. As an example, it may describe a steady-state temperature distribution in plane body Ω, which is thermally anisotropic and inhomogeneous. Where $u(x)$ is an unknown temperature, $a(x)$ is a known variable thermo-conductivity coefficient, $f(x)$ is a known distributed heat source, $\varphi_0(x)$ is the known heat on the displacement boundary, T_a^+ is a surface flux operator where $T_a^+ u(x) = a(x)\frac{\partial u(x)}{\partial n(x)}$, $n(x)$ is the external normal vector to $\partial\Omega$, $\psi_0(x)$ is known heat flux on the boundary.

Theorem 1. *The homogeneous version of BVP (1.7) - (1.9) has only the trivial solution.*

Proof. we follow the idea of the proof for the 3D case in ([CMN09a, Theorem 2.1]). The variational setting for the BVP (1.7) - (1.9) is obtained from the Green's first formula , find $u \in H^1(\Omega)$ with $\gamma^+ u = \varphi_0$ on $\partial_D\Omega$ such that

$$\mathscr{E}_a(u,v) = -\langle f, v\rangle_\Omega + \langle \psi_0, \gamma^+ v\rangle_{\partial_N\Omega},$$

is satisfied for $v \in H^1_0(\Omega,\partial_D\Omega) := \{v \in H^1(\Omega) : \gamma^+ v = 0 \text{ on } \partial_\Omega\}$. Let $u = v$ be a solution of the corresponding homogeneous mixed BVP.

Then the associated variational form is

$$\mathscr{E}_a(u,u) = -\langle f, u\rangle_\Omega + \langle \psi_0, \gamma^+ u\rangle_{\partial_N\Omega} = 0$$

The bi-linear form $\mathscr{E}_a(u,u) : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$ is bounded and is $H^1(\Omega)$-elliptic. Hence,

$$0 = \mathscr{E}_a(u,u) \geq C\|u\|^2_{H^1(\Omega)}$$

is true if and only if $u = 0$.

As real materials are generally never perfectly isotropic their behaviours are modelled by some non-linear and linear PDEs for which the fundamental solution of the "frozen"-coefficient PDE is not available explicitly (as e.g. in the Lamè system of anisotropic elasticity). For such type of problems, as the authors of [AM11] and [Mik05b] showed the need for two operator approach in 3D; which is efficient in handling these problems where the one operator approach fails to work. Hence, we need to define a second operator in order to overcome this problem in 2D. (The one operator approach is investigated in [DM15]).

Let us consider the auxiliary linear elliptic partial differential operator B defined by

$$Bu(x) := B(x,\partial_x)u(x) := \sum_{i=1}^{2} \frac{\partial}{\partial x_i}\left[b(x)\frac{\partial u(x)}{\partial x_i}\right] \tag{1.10}$$

where $b \in C^\infty(\mathbb{R}^2), \quad 0 < b_{min} \leq b(x) \leq b_{max} < \infty, \quad \forall x \in \mathbb{R}^2$. Then for $u \in H^{1,0}(\Omega,\Delta) = H^{1,0}(\Omega,B)$ the associate co-normal derivative operator T_b^+ is defined by (1.5) (and for $u \in H^2(\Omega)$ by (1.4)) with a replaced by b. If $v \in H^{1,0}(\Omega,\Delta), u \in H^1(\Omega)$ then for the operator B holds the first Green identity,

$$\int_\Omega v(x)Bu(x)dx = \int_{\partial\Omega} \gamma^+ v(x)T_b^+ u(x)dS(x) - \int_\Omega \mathscr{E}_b(u,v)dx \tag{1.11}$$

If $u,v \in H^{1,0}(\Omega,\Delta)$, then subtracting (1.11) from (1.5), we obtain the two-operator second Green identity,

$$\begin{aligned}&\int_\Omega \left(u(x)Bv(x) - v(x)Au(x)\right)dx = \\ &\int_{\partial\Omega}\left[\gamma^+ u(x)T_b^+ v(x) - \gamma^+ v(x)T_a^+ u(x)\right]dS(x) - \int_\Omega [a(x)-b(x)]\nabla v(x)\cdot\nabla u(x)dx\end{aligned} \tag{1.12}$$

Note that if $a = b$, then, the last domain integral disappears, and the two-operator Green identity reduces to the classical second Green identity.

1.2 Parametrix based potential operators

A function $P_b(x,y)$ of two variables $x,y \in \mathbb{R}^2$ is called a parametrix (Levi function) for the operator $B(x,\partial_x)$ in $\mathbb{R}^2$ if

$$B(x,\partial_x)P_b(x,y) = \delta(x-y) + R_b(x,y),$$

where $\delta(.)$ is a Dirac-delta distribution and $R_b(x,y)$ is a remainder possessing at most a weak (integrable) singularity at $x=y$. i.e,

$$R_b(x,y) = \mathscr{O}(|x-y|^{-\varkappa}) \text{ with } \varkappa < 3.$$

The parametrix $P_b(x,y)$ and the corresponding remainder for the operator $B(x,\partial_x)$ with singularity at y are given by

$$P_b(x,y) = \frac{1}{2\pi b(y)} \log\left(\frac{|x-y|}{r_0}\right). \tag{1.13}$$

$$R_b(x,y) = \sum_{i=1}^{2} \frac{x_i - y_i}{2\pi b(y)|x-y|^2} \frac{\partial b(x)}{\partial x_i}, \quad x,y \in \mathbb{R}^2. \tag{1.14}$$

Similar to [CMN09a], [ADM17],[AB19] we define the parametrix-based logarithmic and remainder potential operators as:

$$\mathscr{P}_b g(y) = \int_\Omega P_b(x,y)g(x)dx \qquad \mathscr{R}_b g(y) = \int_\Omega R_b(x,y)g(x)dx$$

The single and double layer potential operators based on the parametrix $P_b(x,y)$ are defined as:

$$V_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x \qquad \text{for} \quad y \notin \partial\Omega,$$
$$W_b g(y) := -\int_{\partial\Omega} [T_b^+(x,n(x),\partial_x)P_b(x,y)]g(x)dS_x \qquad \text{for} \quad y \notin \partial\Omega,$$

where g is some scalar function and the integrals are understood in the distributional sense if g is not integrable. The corresponding boundary integral (pseudo-differential) operators of direct surface values of the simple layer potential $\mathscr{V}_b$ and of the double layer potential $\mathscr{W}_b$ for $y \in \partial\Omega$, are

$$\mathscr{V}_b g(y) := -\int_{\partial\Omega} P_b(x,y)g(x)dS_x,$$
$$\mathscr{W}_b g(y) := -\int_{\partial\Omega} [T_b^+(x,n(x),\partial_x)P_b(x,y)]g(x)dS_x.$$

For $y \in \partial\Omega$, the co-normal derivatives associated with the operator A of the simple layer potential and of the double layer potential are given by:

$$T_a^+ V_b g(y) := \frac{a(y)}{b(y)} T_b^+ V_b g(y), \tag{1.15}$$

$$\mathscr{L}_{ab}^+ g(y) := T_a^+ W_b g(y) = \frac{a(y)}{b(y)} T_b^+ W_b g(y) = \frac{a(y)}{b(y)} \mathscr{L}_b^+ g(y), \tag{1.16}$$

The direct value operators associated with (1.15) are

$$\mathscr{W}'_{ab}g(y) = -\int_{\partial\Omega}[T_a^+(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x = \frac{a(y)}{b(y)}\mathscr{W}'_b g(y), \tag{1.17}$$

where

$$\mathscr{W}'_b g(y) = -\int_{\partial\Omega}[T_b^+(y,n(y),\partial_y)P_b(x,y)]g(x)dS_x. \tag{1.18}$$

From equations (1.15)-(1.18) we deduce representations of the parametrix-based surface potential boundary operators in terms of their counterparts for $b=1$, that is, associated with the fundamental solution $P_\Delta = \frac{1}{2\pi}\log\left(\frac{|x-y|}{r_0}\right)$ of the Laplace operator Δ.

$$\mathscr{P}_b = \frac{1}{b}\mathscr{P}_\Delta, \quad \mathscr{R}_b = \frac{1}{b}\sum_{i=1}^{2}\partial_i\mathscr{P}_\Delta\left(g(\partial_i b)\right), \tag{1.19}$$

$$\frac{a}{b}V_a g = V_b g = \frac{1}{b}V_\Delta g, \quad \frac{a}{b}W_a\left(\frac{bg}{a}\right) = W_b g = \frac{1}{b}W_\Delta(bg), \tag{1.20}$$

$$\frac{a}{b}\mathscr{V}_a g = \mathscr{V}_b g = \frac{1}{b}\mathscr{V}_\Delta g, \quad \frac{a}{b}\mathscr{W}_a\left(\frac{bg}{a}\right) = \mathscr{W}_b g = \frac{1}{b}\mathscr{W}_\Delta(bg), \tag{1.21}$$

$$\mathscr{W}'_{ab}g = \frac{a}{b}\mathscr{W}'_b g = \frac{a}{b}\left\{\mathscr{W}'_\Delta(g) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\mathscr{V}_\Delta g\right\}, \tag{1.22}$$

$$\mathscr{L}^+_{ab}g = \frac{a}{b}\mathscr{L}^+_b g = \frac{a}{b}\left\{\mathscr{L}^+_\Delta(bg) + \left[b\frac{\partial}{\partial n}\left(\frac{1}{b}\right)\right]\gamma^+\mathscr{W}_\Delta(bg)\right\}. \tag{1.23}$$

It is taken into account that b and its derivatives are continuous in $\mathbb{R}^2$ and $\mathscr{L}_\Delta(bg) := \mathscr{L}^+_\Delta(bg) = \mathscr{L}^-_\Delta(bg)$ by the Liapunov-Tauber theorem.

The mapping and jump properties of the parametrix-based Logarithmic and surface potentials follow from [CMN09a], [DM15], [AM11].

Theorem 2. *The following operators are continuous*

$$\begin{aligned} V_b &: H^s(\partial\Omega) \to H^{s+\frac{3}{2}}(\Omega) \quad s \in \mathbb{R} \\ W_b &: H^s(\partial\Omega) \to H^{s+\frac{1}{2}}(\Omega) \quad s \in \mathbb{R} \end{aligned}$$

Theorem 3. *Let $s \in \mathbb{R}$. The following pseudo-differential operators are continuous,*

$$\begin{aligned} \mathscr{V}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \\ \mathscr{W}_b &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \\ \mathscr{W}'_{ab} &: H^s(\partial\Omega) \to H^{s+1}(\partial\Omega) \\ \mathscr{L}^+_{ab} &: H^s(\partial\Omega) \to H^{s-1}(\partial\Omega) \end{aligned}$$

Due to the Rellich compact embedding theorem,(see, e.g., [McL00, Theorem 3.27]) and Theorem 3 implies the following assertion.

Theorem 4. *Let $s \in \mathbb{R}$. Let S_1 and S_2 be nonempty, non-intersecting and $\partial\Omega = \bar{S_1} \cup \bar{S_2}$. Then the following operators*

$$r_{S_2}\mathscr{V}_b : \tilde{H}^s(S_1) \to H^s(S_2)$$
$$r_{S_2}\mathscr{W}_b : \tilde{H}^s(S_1) \to H^s(S_2)$$
$$r_{S_2}\mathscr{W}'_{ab} : \tilde{H}^s(S_1) \to H^s(S_2)$$

are compact.

Theorem 5. *Let $diam(\Omega) < r_0$. Then the single layer potential operator $V_b : H^{-\frac{1}{2}}(\partial\Omega) \to H^{\frac{1}{2}}(\partial\Omega)$ is invertible.*

The following well-known jump relations might be useful for further discussions,

Theorem 6. *Let $g_1 \in H^{-\frac{1}{2}}(\partial\Omega)$ and $g_2 \in H^{\frac{1}{2}}(\partial\Omega)$. Then there holds the following jump relation on $\partial\Omega$.*

$$\gamma^+ V_b g_1 = \mathscr{V}_b g_1 \tag{1.24}$$
$$\gamma^+ W_b g_2 = -\frac{1}{2} g_2 + \mathscr{W}_b g_2 \tag{1.25}$$
$$T_a^+ V_b g_1 = \frac{1}{2}\frac{a}{b} g_1 + \mathscr{W}'_{ab} g_1 = \frac{a}{b}\left(\frac{1}{2} g_1 + \mathscr{W}'_b g_1\right) \tag{1.26}$$

Theorem 7. *Let Ω be a bounded open region of $\mathbb{R}^2$ with simply connected, closed, infinitely smooth boundary $\partial\Omega$. The operators:*

$$\mathscr{P}_b : \tilde{H}^s(\Omega) \to H^{s+2}(\Omega), \quad s \in \mathbb{R} \tag{1.27}$$
$$: H^s(\Omega) \to H^{s+2}(\Omega), \quad s > -\frac{1}{2} \tag{1.28}$$
$$\mathscr{R}_b : \tilde{H}^s(\Omega) \to H^{s+1}(\Omega), \quad s \in \mathbb{R} \tag{1.29}$$
$$: H^s(\Omega) \to H^{s+1}(\Omega), \quad s > -\frac{1}{2} \tag{1.30}$$
$$\gamma^+ \mathscr{P}_b : \tilde{H}^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{3}{2} \tag{1.31}$$
$$: H^s(\Omega) \to H^{s+\frac{3}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{1.32}$$
$$\gamma^+ \mathscr{R}_b : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{1.33}$$
$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{1.34}$$
$$T_a^+ \mathscr{P}_b : \tilde{H}^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{1.35}$$

$$: H^s(\Omega) \to H^{s+\frac{1}{2}}(\partial\Omega), \quad s > -\frac{1}{2} \tag{1.36}$$

$$T_a^+ \mathscr{R}_b : \tilde{H}^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2} \tag{1.37}$$

$$: H^s(\Omega) \to H^{s-\frac{1}{2}}(\partial\Omega), \quad s > \frac{1}{2} \tag{1.38}$$

are continuous and the operators

$$\mathscr{R}_b : H^s(\Omega) \to H^s(\Omega), \quad s > -\frac{1}{2} \tag{1.39}$$

$$: H^s(\Omega) \to H^{s,0}(\Omega, A), \quad s > 1 \tag{1.40}$$

$$r_{S_1} \gamma^+ \mathscr{R}_b : H^s(\Omega) \to H^{s-\frac{1}{2}}(S_1), \quad s > -\frac{1}{2} \tag{1.41}$$

$$r_{S_1} T_a^+ \mathscr{R}_b : H^s(\Omega) \to H^{s-\frac{3}{2}}(S_1), \quad s > \frac{1}{2} \tag{1.42}$$

are compact for any non-empty, with an infinitly smooth curve S_1 of $\partial\Omega$.

Theorem 8. *Let S_1 and $\partial\Omega \backslash \overline{S}_1$ be nonempty, open with smooth part of $\partial\Omega$. Then*

$$\mathscr{L}_{ab}^+ + \frac{a}{b}\frac{\partial b}{\partial n}\left(-\frac{1}{2}I + \mathscr{W}_b\right) = \mathscr{L}_{ab}^- + \frac{a}{b}\frac{\partial b}{\partial n}\left(\frac{1}{2}I + \mathscr{W}_b\right) \quad on \;\; \partial\Omega$$

Moreover, the pseudo-differential operator $r_{S_1}\widehat{\mathscr{L}_{ab}} : \tilde{H}^{\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$ where

$$\widehat{\mathscr{L}_{ab}} := \left[\frac{b}{a}\left(-\frac{1}{2}I + \mathscr{W}_b\right)\right] g = \mathscr{L}_{\Delta}(bg) \quad on \;\; \partial\Omega$$

is invertible, while the operators

$$r_{S_1}\left(\frac{b}{a}\mathscr{L}_{ab}^+ - \widehat{\mathscr{L}_{ab}}\right) : \tilde{H}^{\frac{1}{2}}(S_1) \to H^{\frac{1}{2}}(S_1)$$

are bounded and the operators

$$r_{S_1}\left(\frac{b}{a}\mathscr{L}_{ab}^+ - \widehat{\mathscr{L}_{ab}}\right) : \tilde{H}^{\frac{1}{2}}(S_1) \to H^{-\frac{1}{2}}(S_1)$$

are compact.

1.3 The Two operator third Green identity and integral relations

Let $u, v \in H^{1,0}(\Omega; \Delta)$. Then taking the two operator second Green identity (1.12) and replacing $v(x) = P_b(x,y)$ we obtain ,

$$\int_{\Omega}\{u(x)BP_b(x,y)-P_b(x,y)Au(x)\}\,dx =$$
$$\int_{\partial\Omega}\left[\gamma^+u(x)T_b^+P_b(x,y)-\gamma^+P_b(x,y)T_a^+u(x)\right]dS(x)$$
$$-\int_{\Omega}(a(x)-b(x))\,\nabla P_b(x,y)\cdot\nabla u(x)dx \quad (1.43)$$

then by the standard limiting procedures we have the following parametrix-based two operator third green identity for $y\in\Omega$.

$$u+\mathscr{Z}_b u+\mathscr{R}_b u - V_b T_a^+ u + W_b\gamma^+ u = \mathscr{P}_b Au \text{ in } \Omega \quad (1.44)$$

where

$$\mathscr{Z}_b u(y) = -\int_{\Omega}(a(x)-b(x))\,\nabla_x P_b(x,y)\cdot\nabla u(x)dx = \frac{1}{b(y)}\sum_{i=1}^{3}\partial_i\mathscr{P}_{\Delta}\left[(a-b)(\partial_i b)\right](y) \quad (1.45)$$

Note that the above representation(1.44) is Boundary Domain Integro-Differential Equation (BDIDE)

Using the Gauss divergence theorem, we can re-write $\mathscr{Z}_b u(y)$ in the form that does not involve derivative of u

$$\mathscr{Z}_b u(y) = \left(\frac{a(x)}{b(x)}-1\right)u(y)+\widehat{\mathscr{Z}_b}u(y) \quad (1.46)$$

$$\widehat{\mathscr{Z}_b}u(y) = \frac{a(x)}{b(x)}\mathscr{R}_a u(y)-\mathscr{R}_b u(y)+\frac{a(x)}{b(x)}W_b\gamma^+u(y)-W_b\gamma^+u \quad (1.47)$$

which allows to call $\mathscr{Z}_b$ integral operator inspite of its integro-differential representation.

Note that substituting (1.46)-(1.47) in (1.44) and multiplying by $\frac{b(y)}{a(y)}$ one reduces (1.44) to the one-operator parametrix-based third Green identity obtained in [CMN09a],

$$u+\mathscr{R}_a u - V_a T_a^+ u + W_a\gamma^+ u = \mathscr{P}_a Au \quad \text{in } \Omega$$

Relations (1.45), (1.46)-(1.47) and the mapping properties of $\mathscr{P}_a,\mathscr{R}_a,\mathscr{R}_b,W_a$ and W_b imply the following assertion.

Theorem 9. *The operators,*

$$\mathscr{Z}_b : H^s(\Omega)\to H^s(\Omega) \quad \text{for} \quad s>\frac{1}{2}$$

$$\widehat{\mathscr{Z}_b} : H^s(\Omega)\to H^{s,0}(\Omega,\Delta) \quad \text{for} \quad s\geq 1$$

are continuous.

If $u\in H^{1,0}(\Omega;\Delta)$ is a solution of equation (1.7), then (1.44) gives

$$u+Z_b u+\mathscr{R}_b u-V_b T_a^+ u+W_b\gamma^+ u=\mathscr{P}_b f \quad \text{in } \Omega \tag{1.48}$$

Applying the trace and co-normal derivative operator to equation (1.48) and using the jump relations stated on Theorem 6, we obtain

$$u+\mathscr{Z}_b u+\mathscr{R}_b u-V_b T_a^+ u+W_b\gamma^+ u=\mathscr{P}_b f \text{ in } \Omega \tag{1.49}$$

$$\frac{1}{2}\gamma^+ u+\gamma^+\mathscr{Z}_b u+\gamma^+\mathscr{R}_b u-\mathscr{V}_b T_a^+ u+\mathscr{W}_b\gamma^+ u=\gamma^+\mathscr{P}_b f \text{ in } \partial\Omega \tag{1.50}$$

$$\left(1-\frac{a}{2b}\right)T_a^+ u+T_a^+\mathscr{Z}_b u+T_a^+\mathscr{R}_b u-\mathscr{W}_{ab}T_a^+ u+\mathscr{L}_{ab}^+\gamma^+ u=T_a^+\mathscr{P}_b f \text{ in } \partial\Omega \tag{1.51}$$

For some functions f,Ψ and Φ let us consider a more general 'indirect' integral relation associated with equation (1.48).

$$u+\mathscr{Z}_b u+\mathscr{R}_b u-V_b\Psi+W_b\Phi=\mathscr{P}_b f \text{ in } \Omega \tag{1.52}$$

Lemma 1. *Let $f\in L_2(\Omega),\Psi\in H^{-\frac{1}{2}}(\partial\Omega),\Phi\in H^{\frac{1}{2}}(\partial\Omega)$ and $u\in H^1(\Omega)$, satisfy equation (1.52). Then $u\in H^{1,0}(\Omega;\Delta)$ and is a solution of PDE*

$$Au=f \text{ in } \Omega \tag{1.53}$$

and

$$V_b(\Psi-T_a^+ u)-W_b(\Phi-\gamma^+ u)=0, \quad y\in\Omega \tag{1.54}$$

Proof. The proof is same as that of [AM11, Lemma 3.1] and [CMN09a, Lemma 4.1]

Lemma 2. *i) Let $\Psi^*\in H^{-\frac{1}{2}}(\partial\Omega)$, and $diam(\Omega)<r_0$ or $\Psi^*\in H_{**}^{-\frac{1}{2}}(\partial\Omega)$.*
If $V_b\Psi^*=0$ in Ω, then $\Psi^*=0$.
ii) Let $\Phi^*\in H^{\frac{1}{2}}(\partial\Omega)$. If $W_b\Phi^*=0$ in Ω, then $\Phi^*=0$.
iii) Let $\partial\Omega=\overline{S_1}\cup\overline{S_2}$, where S_1,S_2 are non-intersecting and S_1 is nonempty. Let $diam(\Omega)<r_0$ or $\Psi^*\in H_{**}^{-\frac{1}{2}}(\partial\Omega)$ and $\Psi^*\in\tilde{H}^{-\frac{1}{2}}(S_1)$, $\Phi^*\in\tilde{H}^{\frac{1}{2}}(S_2)$. If

$$V_b\Psi^*-W_b\Phi^*=0, \quad y\in\Omega$$

then $\Psi^*=0$ and $\Phi^*=0$ on $\partial\Omega$

Proof. (i.) Let us take the trace of equation (i) on $\partial\Omega$, by the jump relation (1.24) we have

$$\gamma^+V_b\Psi^*=\mathscr{V}_b\Psi^*=0 \text{ on } \partial\Omega.$$

Then the result follows from the invertibility of the single layer potential given in Theorem 5. (ii.)Let us take the trace of equation (ii) on $\partial\Omega$, and use the jump relation (1.25) to obtain,

$$W_b\Phi^*=-\frac{1}{2}\Phi^*+\mathscr{W}_b\Phi^*=0 \quad \text{on} \quad \partial\Omega$$

Multiplying this equation by $b(y)$, denoting $\hat{\Phi}^*=b\Phi^*$ and we obtain equation

$$-\frac{1}{2}\hat{\Phi}^*+\mathscr{W}_\Delta\hat{\Phi}^*=0 \quad \text{on} \quad S$$

Since this equation for Φ^* is uniquely solvable and $b(y) \neq 0$,this implies point (ii).i.e., it has only the trivial solution, see. e.g.([DL90], Chapter XI, Part B, §2, Remark 8)

(iii.)To prove (iii), multiplying the equation by $b(y)$, we have

$$V_\Delta \Psi^* - W_\Delta (b\Phi^*) = 0 \text{ in } \Omega$$

Take the traces of this equation and its co-normal derivative on S_1 and S_2, respectively, to obtain

$$\begin{cases} r_{S_1}\mathscr{V}_\Delta \Psi^* - r_{S_1}\left(-\frac{1}{2}\hat{\Phi}^* + \mathscr{W}_\Delta \hat{\Phi}^*\right) = 0 \quad \text{on} \quad S_1 \\ r_{S_2}\left(\frac{1}{2}\Psi^* + \mathscr{W}'_\Delta \Psi^*\right) - r_{S_2}\mathscr{L}^+_\Delta \hat{\Phi}^* = 0 \quad \text{on} \quad S_2 \end{cases}$$

since $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(S_1)$ and $\Phi^* \in \tilde{H}^{\frac{1}{2}}(S_2)$ follows

$$\begin{cases} r_{S_1}\mathscr{V}_\Delta \Psi^* - r_{S_1}\mathscr{W}_\Delta \hat{\Phi}^* = 0 \quad \text{on} \quad S_1 \\ r_{S_2}\mathscr{W}'_\Delta \Psi^* - r_{S_2}\mathscr{L}^+_\Delta \hat{\Phi}^* = 0 \quad \text{on} \quad S_2 \end{cases} \tag{1.55}$$

where $\hat{\Phi}^* = b\Phi^*$.

The system of equations (1.55) can be written in a matrix form as

$$\mathscr{A}_\Delta \chi = 0$$

where

$$\mathscr{A}_\Delta := \begin{bmatrix} r_{S_1}\mathscr{V}_\Delta & -r_{S_1}\mathscr{W}_\Delta \\ r_{S_2}\mathscr{W}'_\Delta & -r_{S_2}\mathscr{L}^+_\Delta \end{bmatrix}; \qquad \chi = \begin{bmatrix} \Psi^* \\ \hat{\Phi}^* \end{bmatrix}$$

The hypersingular operator $\mathscr{L}^+_\Delta$ is $H^{\frac{1}{2}}_{**}(\partial\Omega)$- elliptic (see e.g., [Ste07, Section 6.6.2, particularly eqn. 6.38]). i.e.,

$$\forall \Phi^* \in H^{\frac{1}{2}}_{**}(\partial\Omega), \quad \langle \mathscr{L}^+_\Delta \Phi^*, \Phi^* \rangle \geq C\|\Phi^*\|^2_{H^{\frac{1}{2}}(\partial\Omega)}$$

Using the norm equivalence Theorem of Sobolev (see [Ste07, Theorem 2.6]), let $S_2 \subset \partial\Omega$ be an open part. For a given $\hat{\Phi}^* \in \tilde{H}^{\frac{1}{2}}(S_2)$, let $\tilde{\Phi}^* \in H^{\frac{1}{2}}(\partial\Omega)$ denote the extension defined by:

$$\tilde{\Phi}^*(x) = \begin{cases} \hat{\Phi}^*(x) \qquad for \quad x \in S_2 \\ 0 \qquad elsewhere \end{cases}$$

As in the norm equivalence Theorem of Sobolev, (see e.g., [Ste07, Theorem 2.6]) we defined

$$\|w\|^2_{H^{\frac{1}{2}}(\partial\Omega),S_2} = \|w\|^2_{L_2(\partial\Omega)\setminus S_2} + |w|^2_{H^{\frac{1}{2}}(\partial\Omega)}$$

to be equivalent norm in $H^{\frac{1}{2}}(\partial\Omega)$. Hence we have for $\hat{\Phi}^* \in \tilde{H}^{\frac{1}{2}}(S_2)$

$$\begin{aligned} \langle \mathscr{L}^+_\Delta \hat{\Phi}^*, \hat{\Phi}^* \rangle_{S_2} &= \langle \mathscr{L}^+_\Delta \tilde{\Phi}^*, \tilde{\Phi}^* \rangle_{S_2} \geq C|\tilde{\Phi}^*|^2_{H^{\frac{1}{2}}(\partial\Omega)} = C\left\{ \|\tilde{\Phi}^*\|^2_{L_2(\partial\Omega)\setminus S_2} + |\tilde{\Phi}^*|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right\} \\ &= C\|\tilde{\Phi}^*\|^2_{H^{\frac{1}{2}}(\partial\Omega),S_2} \geq C\|\tilde{\Phi}^*\|^2_{H^{\frac{1}{2}}(\partial\Omega)} = C\|\hat{\Phi}^*\|^2_{\tilde{H}^{\frac{1}{2}}(S_2)} \end{aligned}$$

which implies that

$$\langle \mathscr{L}^+_\Delta \hat{\Phi}^*, \hat{\Phi}^* \rangle_{S_2} \geq C\|\hat{\Phi}^*\|^2_{\tilde{H}^{\frac{1}{2}}(S_2)} \qquad \text{for} \quad \hat{\Phi}^* \in H^{\frac{1}{2}}_{**}(S_2)$$

and therefore the $\tilde{H}^{\frac{1}{2}}(S_2)$-ellipticity of the hypersingular boundary integral operator $\mathscr{L}^+_\Delta$. In addition, the ellipticity of single layer potential operator $\mathscr{V}_\Delta$ in $\tilde{H}^{-\frac{1}{2}}(S_1)$ follows from Theorem

5 and similarly computed, i.e.,
$\langle \mathscr{V}_\Delta \Psi^*, \Psi^* \rangle_{S_1} \geq c \|\Psi^*\|^2_{\tilde{H}^{-\frac{1}{2}}(S_1)}, \forall \Psi^* \in \tilde{H}^{-\frac{1}{2}}(S_1)$ if and only if $diam(\Omega) < r_0$.
Moreover,

$$r_{S_1}\mathscr{W}_\Delta : \tilde{H}^{\frac{1}{2}}(S_2) \longrightarrow H^{\frac{1}{2}}(S_1)$$
$$r_{S_2}\mathscr{W}'_\Delta : \tilde{H}^{-\frac{1}{2}}(S_1) \longrightarrow H^{-\frac{1}{2}}(S_2)$$

are mutually adjoint, i.e

$$\langle r_{S_1}\mathscr{W}_\Delta \hat{\Phi}^*, \Psi^* \rangle_{S_1} = \langle \hat{\Phi}^*, r_{S_2}\mathscr{W}'_\Delta \Psi^* \rangle_{S_2}$$

for arbitrary $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(S_1)$ and $\hat{\Phi}^* \in \tilde{H}^{\frac{1}{2}}(S_2)$.
Consequently we drive the inequality

$$\langle \mathscr{A}_\Delta \chi, \chi \rangle \geq C \left(\|\Psi^*\|^2_{H^{-\frac{1}{2}}(\partial\Omega)} + \|\hat{\Phi}^*\|^2_{H^{\frac{1}{2}}(\partial\Omega)} \right)$$

Due to (1.55) this implies $\Psi^* = 0$ and $\hat{\Phi}^* = 0$, keeping in mind that $b(y) \neq 0$, we have $\Phi^* = 0$ on $\partial\Omega$, which completes the proof.

Representation Lemma

To prove invertibility of the BDIE operators we need the following representation statements following ([CMN09a], Lemma 5.13).

Lemma 3. *Let $diam(\Omega) < r_0$ or $\Psi \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$, and let $\partial\Omega = \overline{S}_1 \cup \overline{S}_2$, where S_1 and S_2 are nonintersecting curves of $\partial\Omega$. For any triple $\mathscr{F} = (F, \Psi, \Phi)^T \in H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2)$. There exists a unique triple*

$$(f_*, \Psi_*, \Phi_*) = \tilde{\mathscr{C}}_{S_1,S_2}\mathscr{F} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$F = \mathscr{P}_b f_* + V_b \Psi_* - W_b \Phi_* \quad in \quad \Omega \tag{1.56}$$
$$\Psi = r_{S_1}\Psi_* \qquad on \quad S_1 \tag{1.57}$$
$$\Phi = r_{S_2}\Phi_* \qquad on \quad S_2 \tag{1.58}$$

Moreover, the operator

$$\tilde{\mathscr{C}}_{S_1,S_2} : H^{1,0}(\Omega, A) \times H^{-\frac{1}{2}}(S_1) \times H^{\frac{1}{2}}(S_2) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

The cases when $S_1 = \varnothing$ or $S_2 = \varnothing$ need to be considered separately.

Lemma 4. *Let $diam(\Omega) < r_0$ or $\Psi_* \in H^{-\frac{1}{2}}_{**}(\partial\Omega)$. For any function $\mathscr{F}_{\Psi_*} \in H^{1,0}(\Omega, A)$, there exists a unique couple $(f_*, \Psi_*) = \mathscr{C}_\Psi \mathscr{F}_{\Psi_*} \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathscr{F}_{\Psi_*} = \mathscr{P}_b f_* + V_b \Psi_* \quad in \;\; \Omega \tag{1.59}$$

and $\mathscr{C}_\Psi : H^{1,0}(\Omega, A) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ is a bounded linear operator.

Proof. We adapt here the proof scheme from [Mik06, Lemma 5.5].
Suppose first there exist some functions $f_*(y), \Psi_*(y)$ satisfying (1.59) and find their expressions in terms of $\mathscr{F}_{\Psi_*}(y)$. Taking into account definitions for the volume and the single layer potentials, ansatz (1.59) can be rewritten as :

$$b(y)\mathscr{F}_{\Psi_*}(y) = \mathscr{P}_\Delta f_* + V_\Delta(\Psi_*)(y) \qquad y \in \Omega \tag{1.60}$$

Applying the laplace operator to (1.60) we obtain that

$$\Delta(b\mathscr{F}_{\Psi_*})(y) = f_*(y) \qquad y \in \Omega \tag{1.61}$$

Then (1.60) can be rewritten as:

$$V_\Delta(\Psi_*)(y) = Q(y) \qquad y \in \Omega \tag{1.62}$$

where

$$Q(y) = b(y)\mathscr{F}_{\Psi_*}(y) - \mathscr{P}_\Delta[\Delta(b\mathscr{F}_{\Psi_*})](y) \qquad y \in \Omega \tag{1.63}$$

It is easy to check that Q is harmonic functions in Ω as well as (1.61). Then Trace of (1.62) on the boundary gives

$$\gamma^+ V_\Delta(\Psi_*)(y) = \gamma^+ Q(y) \quad \text{implies} \quad \mathscr{V}_\Delta(\Psi_*)(\mathrm{y}) = \gamma^+\mathrm{Q}(\mathrm{y}) \quad \mathrm{y} \in \partial\Omega \tag{1.64}$$

Since $\mathscr{V}_\Delta : H^s(\partial\Omega) \to H^{s+1}(\partial\Omega), s \in \mathbb{R}$ is an isomorphism, (see, e.g., [DL90], chap XI, Part B, sec. 2, Remark 1) and $b(y) \neq 0$ we obtain the following expression of Ψ_*

$$\Psi_*(y) = \mathscr{V}_\Delta^{-1}\gamma^+ Q(y) \quad y \in \partial\Omega \tag{1.65}$$

Relation (1.61) and (1.65) implies uniqueness of the couple f_*, Ψ_* Now we have to prove that $f_*(y), \Psi_*(y)$ given by (1.61) and (1.65) do satisfy (1.59). Indeed, the potential $V_\Delta \Psi_*(y)$ with $\Psi_*(y)$ given by (1.65) is harmonic function, and one can check that Q given by (1.63) is also harmonic. Since (1.64) implies that they coincide on the boundary, the two harmonic functions should also coincide in the domain, i.e, (1.62) holds true, which implies (1.59). Thus, (1.61), (1.65) , (1.63) give

$$(f_*, \Psi_*) = \mathscr{C}_\Psi \mathscr{F}_{\Psi_*} = \left(\Delta(b\mathscr{F}_{\Psi_*}), \mathscr{V}_\Delta^{-1}\gamma^+ Q\right) = \left(\Delta(b\mathscr{F}_{\Psi_*}), \mathscr{V}_\Delta^{-1}\gamma^+[b(y)\mathscr{F}_{\Psi_*} - \mathscr{P}_\Delta[\Delta(b\mathscr{F}_{\Psi_*})]\right)$$

and thus we constructed a bounded operator $C_\Psi : H^{1,0}(\Omega; A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$

Considering a couple $(F, \Phi)^T = \mathscr{F} \in H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega)$ and employing Lemma 4 for $\mathscr{F}_{\Psi_*} = F + W_b\Phi \in H^{1,0}(\Omega, A)$ we arrive at the following statement.

Corollary 1.3.1 *Let* $diam(\Omega) < r_0$ *or* $\Psi \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. *For any couple*

$$(F, \Phi)^T = \mathscr{F} \in H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega)$$

there exists a unique triple

$$(f_*, \Psi_*, \Phi_*)^T = \tilde{\mathscr{C}}_{\Phi_*}\mathscr{F}_* \in L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$F = \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* \quad in \;\; \Omega \tag{1.66}$$

$$\Phi = \Phi_* \quad on \;\; \partial\Omega \tag{1.67}$$

Moreover, the operator

$$\tilde{\mathscr{C}}_{\Phi_*} : H^{1,0}(\Omega, A) \times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega) \times H^{-\frac{1}{2}}(\partial\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Proof. Taking $\Phi_* = \gamma^+ F - \Phi$ and applying Lemma 4 for $\mathscr{F}_{\Psi_*} = F + W_b \Phi \in H^{1,0}(\Omega, A)$ we obtain that the existence of equations (1.66)- (1.67). To prove the uniqueness, we consider its homogeneous case. i.e., with $F = 0$ and $\Phi = 0$. then (1.67) implies $\Phi_* = 0$ and thus by (1.66) and Lemma 4 we also ontain $\Psi_* = 0, f_* = 0$.

Lemma 5. *For any function $\mathscr{F}_{\Phi_*} \in H^{1,0}(\Omega)$ there exists a unique couple $(f_*, \Phi_*) = C_\Phi \mathscr{F}_{\Phi_*} \in L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ such that*

$$\mathscr{F}_{\Phi_*} = \mathscr{P}_b f_* - W_b \Phi_* \qquad \text{in} \quad \Omega \tag{1.68}$$

and $C_\Phi : H^{1,0}(\Omega; A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ is a bounded linear operator.

Proof. Suppose first there exist some functions $f_*(y), \Phi_*(y)$ satisfying (1.68) and find their expressions in terms of $\mathscr{F}_{\Phi_*}(y)$. Taking into account definitions for the volume and the double layer potentials, ansatz (1.68) can be rewritten as :

$$b(y)\mathscr{F}_{\Phi_*}(y) = \mathscr{P}_\Delta f_* - W_\Delta(b\Phi_*)(y) \qquad y \in \Omega \tag{1.69}$$

Applying the Laplace operator to (1.69) we obtain that

$$\Delta(b\mathscr{F}_{\Phi_*})(y) = f_*(y) \tag{1.70}$$

Then (1.69) can be rewritten as:

$$W_\Delta(b\Phi_*)(y) = Q(y) \qquad y \in \Omega \tag{1.71}$$

where

$$Q(y) = \mathscr{P}_\Delta f_* - b(y)\mathscr{F}_{\Phi_*}(y) \qquad y \in \Omega \tag{1.72}$$

It is easy to check that Q is harmonic functions in Ω as well as (1.70). Then Trace of (1.71) on the boundary gives

$$\gamma^+ W_\Delta(b\Phi_*)(y) = \gamma^+ Q(y) \quad \text{implies} \quad \left[-\frac{1}{2}\mathrm{I} + \mathscr{W}_\Delta\right](\mathrm{b}\Phi_*)(\mathrm{y}) = \gamma^+ \mathrm{Q}(\mathrm{y}) \tag{1.73}$$

Since $\left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]$ is an isomorphism, (see, e.g., [DL90], chap XI, Part B, sec. 2, Remark 8) and $b(y) \neq 0$ we obtain the following expression of Φ_*

$$\Phi_*(y) = \frac{1}{b(y)}\left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]^{-1} \gamma^+ Q(y) \quad y \in \partial\Omega \tag{1.74}$$

Now we have to prove that $f_*(y), \Phi_*(y)$ given by (1.70) and (1.74) do satisfy (1.68). Indeed, the potential $W_\Delta(b\Phi_*)(y)$ with $\Phi_*(y)$ given by (1.74) is harmonic function, and one can check that Q given by (1.72) is also harmonic. Since (1.73) implies that they coincide on the boundary, the two harmonic functions should also coincide in the domain, i.e, (1.71) holds true, which implies (1.68). Thus, (1.70), (1.74) , (1.72) give

$$(f_*, \Phi_*) = \mathscr{C}_\Phi \mathscr{F}_{\Phi_*} = \left(\Delta(b\mathscr{F}_{\Phi_*}), \frac{1}{b}\left[-\frac{1}{2}I + \mathscr{W}_\Delta\right]^{-1} \gamma^+ Q\right)$$

and thus we constructed a bounded operator $\mathscr{C}_\Phi : H^{1,0}(\Omega; A) \to L_2(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$

Considering a couple $(F,\Psi)^T \in H^{1,0}(\Omega,A)\times H^{-\frac{1}{2}}(\partial\Omega)$ and employing Lemma 5 for $\mathscr{F}_{\Phi_*} = F - V_b\Psi \in H^{1,0}(\Omega,A)$ we arrive at the following statement.

Corollary 1.3.2 *For any couple*

$$(F,\Psi)^T = \mathscr{F}_* \in H^{1,0}(\Omega,A)\times H^{-\frac{1}{2}}(\partial\Omega)$$

there exists a unique triple

$$(f_*,\Psi_*,\Phi_*)^T = \tilde{\mathscr{C}}_{\Psi_*}\mathscr{F}_* \in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$$

such that

$$F = \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* \quad in \;\; \Omega \tag{1.75}$$

$$\Psi = \Psi_* \quad on \;\; \partial\Omega \tag{1.76}$$

Moreover, the operator

$$\tilde{\mathscr{C}}_{\Psi_*} : H^{1,0}(\Omega,A)\times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega)$$

is linear and continuous.

Proof. Taking $\Psi_* = T_a^+F - \Psi$ and applying Lemma 5 for $\mathscr{F}_* = F - V_b\Phi \in H^{1,0}(\Omega,A)$ we prove the existence of equations (1.75)- (1.76). To prove the uniqueness, we consider its homogeneous case. i.e., with $F=0$ and $\Psi=0$. then (1.76) implies $\Psi_*=0$ and thus by (1.75) and Lemma 5 we also ontain $\Phi_*=0, f_*=0$.

Now consider the original Dirichlet BVP for $u \in H^1(\Omega)$ given on (1.7) -(1.9). We can rewrite the given mixed BVP in a matrix form

$$A^{DN}u = F^{DN}$$

$$\text{where} \quad A^{DN} := \begin{bmatrix} \mathscr{A} \\ \gamma^+ \\ T_a^+ \end{bmatrix}. \qquad F^{DN} := \begin{bmatrix} f \\ \varphi_0 \\ \psi_0 \end{bmatrix}.$$

The operator $A^{DN} : H^{1,0}(\Omega,A) \to L_2(\Omega)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)$ is evidentely continuous and due to the uniqueness theorem for BVP, it is also injective. The following assertions are based on [CMN09a]. The following assertion is well-known and can be proved using variational setting and the Lax-Milgram Lemma.

Theorem 10. *The operator*

$$A^{DN} : H^{1,0}(\Omega,A) \to L_2(\Omega)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega) \tag{1.77}$$

is continuous and continuously invertible.

1.4 Two-operator BDIE systems

Let $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$, be some extensions of the given data $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ from $\partial_D\Omega$ to $\partial\Omega$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ from $\partial_N\Omega$ to $\partial\Omega$ respectively.
Let us denote

$$F_0 := \mathscr{P}_b f + V_b \Psi_0 - W_b \Phi_0 \ \text{ in } \ \Omega \tag{1.78}$$

Note that for $f \in L_2(\Omega), \Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ we have the inclusion $F_0 \in H^{1,0}(\Omega, A)$ due to the mapping properties of the Newtonian and layer potentials. To reduce BVP (1.7)-(1.9) to one or another two-operator BDIE system, we shall use equation (1.49) in Ω, and restrictions of equation (1.50) or (1.51) to appropriate parts of the boundary. We shall always substitute $\Phi_0 + \varphi$ for $\gamma^+ u$ and $\Psi_0 + \psi$ for $T_a^+ u$, where $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$, are considered as known, while ψ belongs to $\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and φ belongs to $\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ due to the boundary conditions (1.8)-(1.9) and are to be found along with $u \in H^{1,0}(\Omega, \Delta)$. This will lead us to segregated BDIE systems with respect to the unknown triple

$$\mathscr{U} := [u, \psi, \varphi]^T \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \tag{1.79}$$

1.4.1 Boundary-Domain Integral Equation system M11

Let us use equation (1.49) in Ω, the restriction of equation (1.50) on $\partial_D\Omega$ and the restriction of equation (1.51) on $\partial_N\Omega$, and using these on (1.44) we get

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b T_a^+ u + W_b \gamma^+ u = \mathscr{P}_b f \ \text{ in } \ \Omega$$

putting $\gamma^+ u := \Phi_0 + \varphi, \ T_a^+ u := \Psi_0 + \psi$ in the above equation will give us

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi \ = \ F_0$$

where $F_0 = \mathscr{P}_b f + V_b \Psi_0 - W_b \Phi_0$. Taking the trace and the co-normal derivative,of the above equation. Then we arrive at the following two-operator segregated system of BDIEs:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0 \ \text{ in } \ \Omega \tag{1.80}$$

$$\gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi + \mathscr{W}_b \varphi = \gamma^+ F_0 - \varphi_0 \ \text{ in } \ \partial_D\Omega \tag{1.81}$$

$$T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} \psi + \mathscr{L}_{ab}^+ \varphi = T_a^+ F_0 - \psi_0 \ \text{ in } \ \partial_N\Omega \tag{1.82}$$

which we call BDIE M11, where M stands for the mixed problem and 11 hints that the integral equations on the Dirichlet and Neumann parts of the boundary are of the first kind. Note that due to Lemma 1 all terms of equation (1.80) belongs to $H^{1,0}(\Omega;\Delta)$ and their co-normal derivatives are well defined.
System (1.80)-(1.82) can be rewritten in the form

$$\mathscr{M}^{11}\mathscr{U} = \mathscr{G}^{11}$$

where

$$\mathscr{M}^{11} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ r_{\partial_D\Omega}\gamma^+[\mathscr{Z}_b + \mathscr{R}_b] & -r_{\partial_D\Omega}\mathscr{V}_b & r_{\partial_D\Omega}\mathscr{W}_b \\ r_{\partial_N\Omega}T_a^+[\mathscr{Z}_b + \mathscr{R}_b] & -r_{\partial_N\Omega}\mathscr{W}'_{ab} & r_{\partial_N\Omega}\mathscr{L}_{ab}^+ \end{bmatrix} \quad \mathscr{G}^{11} := \begin{bmatrix} F_0 \\ r_{\partial_D\Omega}\gamma^+ F_0 - \varphi_0 \\ r_{\partial_N\Omega}T_a^+ F_0 - \psi_0 \end{bmatrix} \tag{1.83}$$

Due to the mapping properties of $V_b, \mathscr{V}_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, \gamma^+\mathscr{R}_b, T_a^+\mathscr{R}_b, \mathscr{Z}_b, \gamma^+\mathscr{Z}_b$, and $\mathrm{T_a^+}\mathscr{Z}_\mathrm{b}$ we have $\mathscr{G}^{11} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \times H^{-\frac{1}{2}}(\partial\Omega)$ and the operator

$$\mathscr{M}^{11} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial_D\Omega) \times H^{-\frac{1}{2}}(\partial_N\Omega) \tag{1.84}$$

is continuous.

Remark 2. $\mathscr{G}^{11} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

1.4.2 Boundary-Domain Integral Equation system M12

To obtain another system, we use equation (1.49) in Ω, and equation (1.50) on the whole boundary $\partial\Omega$. Putting $\gamma^+ u := \Phi_0 + \varphi,\ \ T_a^+ u := \Psi_0 + \psi$ and taking the trace of the above equation. Then we arrive at the following two-operator segregated system of BDIE system M12:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0 \quad \text{in} \quad \Omega \tag{1.85}$$

$$\frac{1}{2}\varphi + \gamma^+ \mathscr{Z}_b u + \gamma^+ \mathscr{R}_b u - \mathscr{V}_b \psi + \mathscr{W}_b \varphi = \gamma^+ F_0 - \Phi_0 \quad \text{on} \quad \partial\Omega \tag{1.86}$$

System (1.85)-(1.86) can be rewritten in the form

$$\mathscr{M}^{12} \mathscr{U} = \mathscr{G}^{12}$$

where

$$\mathscr{M}^{12} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ \gamma^+[\mathscr{Z}_b + \mathscr{R}_b] & -\mathscr{V}_b & \frac{1}{2} I + \mathscr{W}_b \end{bmatrix} \qquad \mathscr{G}^{12} := \begin{bmatrix} F_0 \\ \gamma^+ F_0 - \Phi_0]^T \end{bmatrix} \tag{1.87}$$

Due to the mapping properties of $V_b, \mathscr{V}_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, \gamma^+ \mathscr{R}_b, \mathscr{Z}_b$, and $\gamma^+ \mathscr{Z}_\mathrm{b}$ we have $\mathscr{G}^{12} \in H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega)$ and the operator

$$\mathscr{M}^{12} : H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega) \to H^1(\Omega) \times H^{\frac{1}{2}}(\partial\Omega) \tag{1.88}$$

is continuous.

Remark 3. Let $diam(\Omega) < r_0$ or $\Psi_0 \in H_{**}^{-\frac{1}{2}}(\partial\Omega)$. Then $\mathscr{G}^{12} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

1.4.3 Boundary-Domain Integral Equation system M21

To obtain another system, we use equation (1.49) in Ω, and equation (1.51) on the whole boundary $\partial\Omega$. Putting $\gamma^+ u := \Phi_0 + \varphi, \quad T_a^+ u := \Psi_0 + \psi$ and taking the co-normal derivative of the above equation. Then we arrive at the following two-operator segregated system of BDIE system M21:

$$u + \mathscr{Z}_b u + \mathscr{R}_b u - V_b \psi + W_b \varphi = F_0 \qquad \text{in} \quad \Omega \tag{1.89}$$

$$\left(1 - \frac{a}{2b}\right)\psi + T_a^+ \mathscr{Z}_b u + T_a^+ \mathscr{R}_b u - \mathscr{W}'_{ab} \psi + \mathscr{L}_{ab}^+ \varphi = T_a^+ F_0 - \Psi_0 \quad \text{on} \quad \partial\Omega \tag{1.90}$$

System (1.89)-(1.90) can be rewritten in the form

$$\mathscr{M}^{21} \mathscr{U} = \mathscr{F}^{21}$$

where

$$\mathscr{M}^{21} := \begin{bmatrix} I + \mathscr{Z}_b + \mathscr{R}_b & -V_b & W_b \\ T_a^+[\mathscr{Z}_b + \mathscr{R}_b] & \left(1 - \frac{a}{2b}\right) I - \mathscr{W}'_{ab} & \mathscr{L}_{ab}^+ \end{bmatrix} \qquad \mathscr{G}^{21} := \begin{bmatrix} F_0 \\ T_a^+ F_0 - \Phi_0 \end{bmatrix} \tag{1.91}$$

Due to the mapping properties of $V_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, T_a^+\mathscr{R}_b, \mathscr{Z}_b$, and $\mathrm{T_a^+}\mathscr{Z}_\mathrm{b}$ we have $\mathscr{G}^{21} \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)$ and the operator

$$\mathscr{M}^{21} : H^1(\Omega)\times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega) \tag{1.92}$$

is continuous.

Remark 4. $\mathscr{G}^{21} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

1.4.4 Boundary-Domain Integral Equation system M22

To reduce (1.49)-(1.51) to a BDIE system of almost the second kind (up to the spaces), we use equation (1.49) in Ω, the restriction of equation (1.51) to $\partial_D\Omega$, and the restriction of equation (1.50) to $\partial_N\Omega$. Putting $\gamma^+u := \Phi_0+\varphi$, $T_a^+u := \Psi_0+\psi$ and first taking the co-normal derivative and then the trace of the above equation. Then we arrive at the following two-operator segregated system of BDIE system M22:

$$u+\mathscr{Z}_b u+\mathscr{R}_b u - V_b\psi + W_b\varphi = F_0 \quad \text{in} \quad \Omega \tag{1.93}$$

$$\left(1-\frac{a}{2b}\right)\psi + T_a^+\mathscr{Z}_b u + T_a^+\mathscr{R}_b u - \mathscr{W}'_{ab}\psi + \mathscr{L}^+_{ab}\varphi = T_a^+F_0 - \Psi_0 \quad \text{on} \quad \partial_D\Omega \tag{1.94}$$

$$\frac{1}{2}\varphi + \gamma^+\mathscr{Z}_b u + \gamma^+\mathscr{R}_b u - \mathscr{V}_b\psi + \mathscr{W}_b\varphi = \gamma^+F_0 - \Phi_0 \quad \text{in} \quad \partial_N\Omega \tag{1.95}$$

System (1.93)-(1.95) can be rewritten in the form

$$\mathscr{M}^{22} := \begin{bmatrix} I+\mathscr{Z}_b+\mathscr{R}_b & -V_b & W_b \\ r_{\partial_D\Omega}T^+[\mathscr{Z}_b+\mathscr{R}_b] & \left(1-\frac{a}{2b}\right)I - r_{\partial_D\Omega}\mathscr{W}'_{ab} & r_{\partial_D\Omega}\mathscr{L}^+_{ab} \\ r_{\partial_N\Omega}\gamma^+[\mathscr{Z}_b+\mathscr{R}_b] & -r_{\partial_N\Omega}\mathscr{V}_b & \frac{1}{2}I + r_{\partial_N\Omega}\mathscr{W}_b \end{bmatrix}$$

$$\mathscr{G}^{22} := \begin{bmatrix} F_0 \\ r_{\partial_D\Omega}T_a^+F_0 - \Psi_0 \\ r_{\partial_N\Omega}\gamma^+F_0 - \Phi_0 \end{bmatrix}$$

Due to the mapping properties of $V_b, \mathscr{V}_b, W_b, \mathscr{W}_b, \mathscr{P}_b, \mathscr{R}_b, \gamma^+\mathscr{R}_b, T_a^+\mathscr{R}_b, \mathscr{Z}_b, \gamma^+\mathscr{Z}_b$, and $\mathrm{T_a^+}\mathscr{Z}_\mathrm{b}$ we have $\mathscr{G}^{22} \in H^1(\Omega)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega)$ and the operator

$$\mathscr{M}^{22} : H^1(\Omega)\times \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times \tilde{H}^{\frac{1}{2}}(\partial_N\Omega) \to H^1(\Omega)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega) \tag{1.96}$$

is continuous.

Remark 5. $\mathscr{G}^{22} = 0$ if and only if $(f, \Phi_0, \Psi_0) = 0$.

1.5 Equivalence and Invertibility

Theorem 11. *Let $f \in L_2(\Omega)$ and $\Phi_0 \in H^{\frac{1}{2}}(\partial\Omega)$ and $\Psi_0 \in H^{-\frac{1}{2}}(\partial\Omega)$ be some fixed extensions of $\varphi_0 \in H^{\frac{1}{2}}(\partial_D\Omega)$ and $\psi_0 \in H^{-\frac{1}{2}}(\partial_N\Omega)$ respectively.*

i. If some $u \in H^1(\Omega)$ solves the mixed BVP(1.7)-(1.9) in Ω, then the solution is unique and the triple $(u,\psi,\varphi)^T \in H^1(\Omega)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ where

$$\psi = T_a^+ u - \Psi_0, \quad \varphi = \gamma^+ u - \Phi_0 \quad \text{on } \partial\Omega \tag{1.97}$$

solves the BDIE system M11, M12, M21 and M22.

ii. *Assume diam(Ω) $< r_0$, if a triple $(u,\psi,\varphi) \in H^1(\Omega)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves BDIE system M11, M12, M21 and M22, then the solution is unique, the function u solves BVP(1.8)-(1.9), and relation (1.97) holds.*

Proof. i. Let $u \in H^1(\Omega)$ be a solution of BVP (1.7)-(1.9). Then by Theorem 1 (see, e.g., [CMN09a, Theorem 2.1]), it is unique. Next, set $\varphi = \gamma^+ u - \Phi_0, \quad \psi = T_a^+ u - \Psi_0$. Evidently, $\psi \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$ and $\varphi \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. And recalling how The BDIE system M11, M12, M21 and M22 were constracted, we obtain that the triple $(u,\psi,\varphi)^T$ satisfies the BDIE system M11, M12, M21 and M22. Which completes the proof of item (i).
We give the proofs of item (ii) for the four BDIE systems M11, M12, M21 and M22 independently.

ii. **M11**: Let the triple $(u,\psi,\varphi)^T \in H^1(\Omega)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves the BDIE system M11. Let us consider the trace of the equation (1.80) on $\partial_D\Omega$ taking into account the jump properties (1.24) - (1.25), and subtruct equation (1.81) to obtain

$$r_{\partial_D\Omega}\gamma^+ u = \varphi_0 \qquad \text{on} \quad \partial_D\Omega \tag{1.98}$$

i.e., u satisfies the Dirichlet condition i.e., (1.8).
Taking the co-normal derivative T_a^+ of (1.80) on $\partial_N\Omega$, again with account of the jump properties (1.26), and subtracting equation (1.82), we obtain:

$$r_{\partial_N\Omega} T_a^+ u = \psi_0 \qquad \text{on} \quad \partial_N\Omega \tag{1.99}$$

i.e., u satisfies the Neumann condition i.e., (1.9).
Taking into account that $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D\Omega$ and $\psi = 0, \Psi_0 = \psi_0$ on $\partial_N\Omega$ equation (1.98) and (1.99) imply that the first equation of (1.97) is satisfied on $\partial_N\Omega$ and the second equation of (1.97) satisfied on $\partial_D\Omega$. Equation (1.80) and Lemma 1 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (1.7) and

$$V_b\Psi^* - W_b\Phi^* = 0 \qquad \text{in} \quad \Omega$$

where $\Psi^* = \Psi_0 + \psi - T_a^+ u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$. Since the first equation of (1.97) on $\partial_N\Omega$ and the second equation of (1.97) on $\partial_D\Omega$ already proved, we have $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)$; $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$. Then Lemma 2 (iii.) with

$$S_1 = \partial_D\Omega; S_2 = \partial_N\Omega \qquad \text{implies that } \Psi = \Phi = 0$$

which completes the proof of condition (1.97).

M12 Let $(u,\psi,\varphi) \in H^1(\Omega)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)$ solves the BDIE (1.85)-(1.86). Consider the trace of the equation (1.85) on $\partial\Omega$, taking into account the jump properties (1.24) -(1.25), and subtruct equation (1.86) to obtain

$$\gamma^+ u = \Phi_0 + \varphi \quad \text{on} \quad \partial\Omega \tag{1.100}$$

this means that the second equation of (1.97) holds. Since $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D\Omega$ we see that the Dirichlet condition (1.8) is satisfied. Equation (1.85) and Lemma 1 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (1.7) and

$$V_b\left(\Psi_0 + \psi - T_a^+ u\right) - W_b\left(\Phi_0 + \varphi - \gamma^+ u\right) = 0 \qquad \text{in} \quad \Omega \tag{1.101}$$

Due to (1.100) the second term in (1.101) vanishes, and by Lemma 2(i) we obtain:

$$\Psi_0 + \psi - T_a^+ u = 0 \qquad \text{on} \quad \partial\Omega \tag{1.102}$$

i.e., the first equation in (1.97) is satisfied as well. Since $\psi = 0, \Psi_0 = \psi_0$ on $\partial_N \Omega$ equation (1.102) implies that u satisfies the Neumann Boundary Condition (1.9).

M21 Let $(u, \psi, \varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega)$ solves the BDIE (1.89)-(1.90). Taking the co-normal derivative T_a^+ of the equation (1.89) on $\partial\Omega$, taking into account the jump properties (1.26), and subtruct equation (1.90) we obtain

$$T_a^+ u - \psi = \Psi_0 \qquad \text{on} \quad \partial\Omega \tag{1.103}$$

which proves the first equation of (1.97). Since $\psi = 0, \Psi_0 = \psi_0$ on $\partial_N \Omega$ we see that the Neumann condition (1.9) is satisfied. Equation (1.89) and Lemma 1 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (1.7) and

$$V_b \left(\Psi_0 + \psi - T_a^+ u \right) - W_b \left(\Phi_0 + \varphi - \gamma^+ u \right) = 0 \qquad \text{in} \quad \Omega \tag{1.104}$$

Due to (1.103) the first term in (1.104) vanishes, and by Lemma 2(ii), we obtain:

$$\Phi_0 + \varphi - \gamma^+ u = 0 \qquad \text{on} \quad \partial\Omega \tag{1.105}$$

Which means that the second term in (1.97) holds as well. Taking into account $\varphi = 0, \Phi_0 = \varphi_0$ on $\partial_D \Omega$ equation (1.102) implies that u satisfies the Dirichlet Boundary Condition (1.8).

M22 Let $(u, \psi, \varphi) \in H^1(\Omega) \times \tilde{H}^{-\frac{1}{2}}(\partial_D \Omega) \times \tilde{H}^{\frac{1}{2}}(\partial_N \Omega)$ solves the BDIE (1.93)-(1.95). Taking the co-normal derivative of the equation (1.95) on $\partial_D \Omega$, taking into account the jump properties (1.26), and subtruct equation (1.94) we obtain

$$\psi = T_a^+ u - \Psi_0 \qquad \text{on} \quad \partial_D \Omega \tag{1.106}$$

Further, take the trace of equation (1.93) on $\partial_N \Omega$ and subtract it from (1.95), we get

$$\varphi = \gamma^+ u - \Phi_0 \qquad \text{on} \quad \partial_N \Omega \tag{1.107}$$

These equations imply that the first equation of (1.97) is satisfied on $\partial_D \Omega$ and the second equation of (1.97) is satisfied on $\partial_N \Omega$
Equation (1.93) and Lemma 1 with $\Psi = \psi + \Psi_0; \Phi = \varphi + \Phi_0$ imply that u is a solution to (1.7) and

$$V_b \Psi^* - W_b \Phi^* = 0 \qquad \text{in} \quad \Omega$$

where $\Psi^* = \Psi_0 + \psi - T_a^+ u$ and $\Phi^* = \Phi_0 + \varphi - \gamma^+ u$
Due to (1.97) and (1.93) we have $\Psi^* \in \tilde{H}^{-\frac{1}{2}}(\partial_N \Omega)$, $\Phi^* \in \tilde{H}^{\frac{1}{2}}(\partial_D \Omega)$ and by Lemma 2 (iii) with

$$S_1 = \partial_N \Omega; S_2 = \partial_D \Omega \ \text{ implies that } \ \Psi^* = \Phi^* = 0$$

which completes the proof of condition (1.97) on the whole boundary $\partial\Omega$. Taking into account that $\varphi = 0$ on $\partial_D \Omega$ and $\Phi_0 = \varphi_0$ on $\partial_D \Omega$ and $\psi = 0$ on $\partial_N \Omega$ and $\Psi_0 = \psi_0$ on $\partial_N \Omega$, equation (1.97) imply the Boundary Conditions (1.8-)(1.9).

Unique solvability of the BDIE systems M11, M12, M21 and M22 then follows from the already proved relations (1.97) and the unique solvability of BVP (1.7)-(1.9) stated in item (i).

The mapping properties of operators in (1.27), (1.29), (1.33), (1.35) and (1.37) and Theorem 11 imply the following statement.

Corollary 1.5.1 *The operators*

$$\mathscr{M}^{11}: H^{1,0}(\Omega,A)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega) \quad (1.108)$$

$$\mathscr{M}^{12}: H^{1,0}(\Omega,A)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial\Omega) \quad (1.109)$$

$$\mathscr{M}^{21}: H^{1,0}(\Omega,A)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega,A)\times H^{-\frac{1}{2}}(\partial\Omega) \quad (1.110)$$

$$\mathscr{M}^{22}: H^{1,0}(\Omega,A)\times\tilde{H}^{-\frac{1}{2}}(\partial_D\Omega)\times\tilde{H}^{\frac{1}{2}}(\partial_N\Omega)\to H^{1,0}(\Omega,A)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega) \quad (1.111)$$

are continuous and injective.

Now we are in the position to analyse the invertibility of the operators $\mathscr{M}^{11}$, $\mathscr{M}^{12}$, $\mathscr{M}^{21}$ and $\mathscr{M}^{22}$.

Theorem 12. *The Operators (1.108)- (1.111) are continuously invertible.*

Proof. To prove the invertibility of operator (1.108), let us consider BDIE system M11 with an arbitrary right hand side $\mathscr{F}^{11}=\left(\mathscr{F}_1^{11},\mathscr{F}_2^{11},\mathscr{F}_3^{11}\right)^T\in H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)$. Taking $S_1=\partial_N\Omega, S_2=\partial_D\Omega$ and

$$F=\mathscr{F}_1^{11},\ \ \Psi=r_{\partial_N\Omega}T_a^+\mathscr{F}_1^{11}-\mathscr{F}_3^{11},\ \ \ \Phi=r_{\partial_D\Omega}\gamma^+\mathscr{F}_1^{11}-\mathscr{F}_2^{11}$$

From Lemma 3,(see, e.g., [CMN09a, Lemma 5.13]), $\mathscr{F}^{11}$ can be represented as

$$\begin{aligned}\mathscr{F}_1^{11} &= \mathscr{P}_b f_*+V_b\Psi_*-W_b\Phi_* \quad \text{in} \quad \Omega\\ \mathscr{F}_2^{11} &= r_{\partial_D\Omega}[\gamma^+\mathscr{F}_1^{11}-\Phi_*] \quad \text{on} \quad \partial_D\Omega\\ \mathscr{F}_3^{11} &= r_{\partial_N\Omega}[T_a^+\mathscr{F}_1^{11}-\Psi_*] \quad \text{on} \quad \partial_N\Omega\end{aligned}$$

where the triple

$$(f_*,\Psi_*,\Phi_*)^T=\mathscr{C}_{\partial_N\Omega,\partial_D\Omega}\mathscr{F}^{11}\in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \quad (1.112)$$

is unique and the operator

$$\mathscr{C}_{\partial_N\Omega,\partial_D\Omega}: H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)\to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \quad (1.113)$$

is linear and continuous.
Applying Theorem 11 with

$$f=f_*,\ \ \Psi_0=\Psi_*,\ \ \Phi_0=\Phi_*,\ \ \psi_0=r_{\partial_N\Omega}\Psi_0,\ \ \varphi_0=r_{\partial_D\Omega}\Phi_0 \quad (1.114)$$

we obtain that the system M11 is uniquely solvable and its solution is

$$\mathscr{U}_1=u=(A^{DN})^{-1}\left(f_*,r_{\partial_D\Omega}\Phi_*,r_{\partial_N\Omega}\Psi_*\right)^T,\quad \mathscr{U}_2=\psi=T_a^+\mathscr{U}_1-\Psi_*,\quad \mathscr{U}_3=\varphi=\gamma^+\mathscr{U}_1-\Phi_* \quad (1.115)$$

While $r_{\partial_N\Omega}\mathscr{U}_2=0; r_{\partial_D\Omega}\mathscr{U}_3=0$. Here, by Theorem 10, $(A^{DN})^{-1}$ is the continuous inverse operator to the left-hand-side operator of the mixed BVP (1.7)-(1.9), $A^{DN}: H^{1,0}(\Omega,A)\to H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial_D\Omega)\times H^{-\frac{1}{2}}(\partial_N\Omega)$ (see, e.g., [CMN09a, Corollary 5.16]), Representation (1.112), and continuity of operator (1.113) complete the proof for $\mathscr{M}^{11}$.

To prove the invertibility of operator (1.109), let us consider BDIE system M12 with an arbitrary right hand side $\mathscr{F}^{12}=\left(\mathscr{F}_1^{12},\mathscr{F}_2^{12}\right)^T\in H^{1,0}(\Omega,A)\times H^{\frac{1}{2}}(\partial\Omega)$. Taking

$$\begin{aligned}F&=\mathscr{F}_1^{12}, && \text{in} \quad \Omega\\ \Phi&=\gamma^+\mathscr{F}_1^{12}-\mathscr{F}_2^{12} && \text{on} \ \ \partial\Omega\end{aligned}$$

in Corollary 1.3.1, we obtain the representation

$$\begin{aligned}\mathscr{F}_1^{12} &= \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* \quad \text{in} \quad \Omega\\ \mathscr{F}_2^{12} &= \gamma^+\mathscr{F}_1^{12} - \Phi_* \qquad \text{on} \quad \partial\Omega\end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*)^T = \tilde{\mathscr{C}}_{\Phi_*}\mathscr{F}^{12} \in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{1.116}$$

is unique and the operator

$$\tilde{\mathscr{C}}_{\Phi_*} : H^{1,0}(\Omega, A)\times H^{\frac{1}{2}}(\partial\Omega) \to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{1.117}$$

is linear and continuous.
Applying Theorem 11 with (1.114) i.e.,

$$f = f_*,\;\; \Psi_0 = \Psi_*,\;\; \Phi_0 = \Phi_*,\;\; \psi_0 = r_{\partial_N\Omega}\Psi_0,\;\; \varphi_0 = r_{\partial_D\Omega}\Phi_0$$

we obtain that the system M12 is uniquely solvable and its solution is given by (1.115) which is:

$$\mathscr{U}_1 = u = (A^{DN})^{-1}\left(f_*, r_{\partial_D\Omega}\Phi_*, r_{\partial_N\Omega}\Psi_*\right)^T,\quad \mathscr{U}_2 = \psi = T_a^+\mathscr{U}_1 - \Psi_*,\quad \mathscr{U}_3 = \phi = \gamma^+\mathscr{U}_1 - \Phi_* \tag{1.118}$$

Representation (1.116), and continuity of operator (1.117) complete the proof for $\mathscr{M}^{12}$.

To prove the invertibility of operator (1.110), let us consider BDIE system M21 with an arbitrary right hand side $\mathscr{F}^{21} = \left(\mathscr{F}_1^{21}, \mathscr{F}_2^{21}\right)^T \in H^{1,0}(\Omega, A)\times H^{-\frac{1}{2}}(\partial\Omega)$.
Taking

$$F = \mathscr{F}_1^{21} \quad \text{in} \quad \Omega, \quad \Psi = T_a^+\mathscr{F}_1^{21} - \mathscr{F}_2^{21} \quad \text{on} \quad \partial\Omega$$

following Corollary 1.3.1, we obtain that

$$\begin{aligned}\mathscr{F}_1^{21} &= \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* \quad \text{in} \;\; \Omega\\ \mathscr{F}_2^{21} &= T_a^+\mathscr{F}_1^{21} - \Psi_* \quad \text{on} \;\; \partial\Omega\end{aligned}$$

where the triple

$$(f_*, \Psi_*, \Phi_*) = \tilde{\mathscr{C}}_{\Psi_*}\mathscr{F}^{12} \in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{1.119}$$

is unique and the operator

$$\tilde{\mathscr{C}}_{\Psi_*} : H^{1,0}(\Omega, A)\times H^{-\frac{1}{2}}(\partial\Omega) \to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{1.120}$$

is linear and continuous.
Applying Theorem 11 with substitutions (1.114) i.e.,

$$f = f_*,\;\; \Psi_0 = \Psi_*,\;\; \Phi_0 = \Phi_*,\;\; \psi_0 = r_{\partial_N\Omega}\Psi_0,\;\; \varphi_0 = r_{\partial_D\Omega}\Phi_0$$

we obtain that the system M21 is uniquely solvable and its solution is given by (1.115) i.e.,

$$\mathscr{U}_1 = u = (A^{DN})^{-1}\left(f_*, r_{\partial_D\Omega}\Phi_*, r_{\partial_N\Omega}\Psi_*\right)^T,\quad \mathscr{U}_2 = \psi = T_a^+\mathscr{U}_1 - \Psi_*,\quad \mathscr{U}_3 = \varphi = \gamma^+\mathscr{U}_1 - \Phi_* \tag{1.121}$$

Representation (1.119), and continuity of operator (1.120) complete the proof for $\mathscr{M}^{21}$.

To prove the invertibility of operator (1.111), we apply similar argument as (1.108). Let us consider BDIE system M22 with an arbitrary right hand side $\mathscr{F}^{22} = \left(\mathscr{F}_1^{22}, \mathscr{F}_2^{22}, \mathscr{F}_3^{22}\right)^T \in H^{1,0}(\Omega, A)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega)$ Taking $S_1 = \partial_D\Omega, S_2 = \partial_N\Omega$ and

$$F=\mathscr{F}_1^{22},\ \ \Psi=r_{\partial_D\Omega}T_a^+\mathscr{F}_1^{22}-\mathscr{F}_2^{22},\ \ \ \Phi=r_{\partial_N\Omega}\gamma^+\mathscr{F}_1^{22}-\mathscr{F}_3^{22}$$

in [CMN09a, Lemma 5.13], , presented as Lemma 3, we obtain that $\mathscr{F}^{22}$ can be represented as

$$\begin{aligned}\mathscr{F}_1^{22} &= \mathscr{P}_b f_* + V_b\Psi_* - W_b\Phi_* \quad \text{in} \quad \Omega\\ \mathscr{F}_2^{22} &= r_{\partial_D\Omega}[T_a^+\mathscr{F}_1^{22}-\Psi_*] \quad \text{on} \quad \partial_D\Omega\\ \mathscr{F}_3^{22} &= r_{\partial_N\Omega}[\gamma^+\mathscr{F}_1^{22}-\Phi_*] \quad \text{on} \quad \partial_N\Omega\end{aligned}$$

where the triple

$$(f_*,\Psi_*,\Phi_*)=\mathscr{C}_{\partial_D\Omega,\partial_N\Omega}\mathscr{F}^{22}\in L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{1.122}$$

is unique and the operator

$$\mathscr{C}_{\partial_N\Omega,\partial_D\Omega}:H^{1,0}(\Omega,A)\times H^{-\frac{1}{2}}(\partial_D\Omega)\times H^{\frac{1}{2}}(\partial_N\Omega)\to L_2(\Omega)\times H^{-\frac{1}{2}}(\partial\Omega)\times H^{\frac{1}{2}}(\partial\Omega) \tag{1.123}$$

is linear and continuous.
Applying Theorem 11 with the same substisution as (1.114) we obtain that the system M22 is uniquely solvable and its solution is given by (1.115) i.e.,

$$\mathscr{U}_1=u=(A^{DN})^{-1}\left(f_*,r_{\partial_D\Omega}\Phi_*,r_{\partial_N\Omega}\Psi_*\right)^T,\quad \mathscr{U}_2=\psi=T_a^+\mathscr{U}_1-\Psi_*,\quad \mathscr{U}_3=\varphi=\gamma^+\mathscr{U}_1-\Phi_* \tag{1.124}$$

Representation (1.5), and continuity of operator (1.121) complete the proof for $\mathscr{M}^{22}$.

1.6 Conclusion

In this paper, we have considered the interior Mixed problem for variable coefficient PDE in a two-dimensional domain, where the right hand side function is from $L_2(\Omega)$ and the Dirichlet data from the space $H^{\frac{1}{2}}(\partial_D\Omega)$ and the Neumann data from the space $H^{-\frac{1}{2}}(\partial_N\Omega)$. The BVP was reduced to four systems of Two operator Boundary-Domain Integral Equations and their equivalence to the original BVP was shown. The invertibility of associated boundary domain integral operators in the corresponding Sobolev spaces was also proved. In a similar way, one can consider also the 2D versions of the Two operator BDIEs for the mixed problem in exterior domains, united two operator BDIEs as well as the localized two operator BDIEs can be investigated.

References

[AM11] T. G. Ayele, S. E. Mikhailov: Analysis of two-operator boundary-domain integral equations for variable-coefficient mixed BVP. In: *Eurasian Mathematical Journal (ISSN 2077-9879), Vol. 2, No 3, 2011, 20-41*.

[AB19] Ayele, T.G. and Bekele, S.T.: Two-operator Boundary-Domain Integral Equations for Variable-Coefficient Drichlet BVP in 2D. In: *Methods in Science and Engineering: Computational and Analytic Aspects, C. Constanda and P. Haris (eds.),Birkhäuser, Brighton (2018)*, pp. 53-66.

[DM15] Dufera, T.T. and Mikhailov, S.E.: Analysis of Boundary-Domain Integral Equations-for Variable-Coefficient Drichlet BVP in 2D. In: *Integral Methods in Science and Engineering: Computational and Analytic Aspects, C. Constanda and A. Kirsch (eds.),Birkhäuser, Boston (2015)*, pp. 163-175.

[Gri85] P. Grisvard, *Elliptic Problems in Nonsmooth Domains,* Pitman, Boston–London–Melbourne,1985.

[HCW08] G. C. Hsiao and W. L. Wendland. *Boundary Integral Equation.* Springer and Wiley, Berlin, 2008.

[CMN09a] O. Chakuda, S.E. Mikhailov, D. Natroshvili , Analysis of direct boundary-domain integral equations for a mixed BVP with variable coefficent. I: Equivalence and Invertibility. *J. Integral Equat.and Appl. 21 (2009)*, 499-543.

[Cos88] M. Costabel, Boundary integral operators on Lipschiz domains: elementary results. *SIAM journal on Mathematical Analysis 19 (1988),* 613-626.

[McL00] W. McLean, *Strongly Elliptic Systems and Boundary Integral Equations.* Cambrige University Press, Cambrige, 2000.

[Mik02] S.E. Mikhailov, Localized boundary-domain integral formulations for problems with variable coefficients, *Int. J. Engineering Analysis with Boundary Elements 26 (2002),* 681-690.

[Ste07] O. Steinbach, *Numerical Approximation Methods for Elliptic Boundary Value Problems: Finite and Boundary Elements.* Springer, 2007.

[ADM17] T. G. Ayele, T. T. Dufera and S. E. Mikhailov: Analysis of Boundary-Domain Integral Equationsfor Variable-Coefficient Neumann BVP in 2D. In: *Integral Methods in Science and Engineering: Computational and Analytic Aspects, C. Constanda and A. Kirsch (eds.),Birkhäuser, Boston (2017),* Chapter 3, 21-33.

[CMN11] O. Chkadua, S.E. Mikhailov, D. Natroshvili, Analysis of segregated boundary-domain integral equations for variable-coefficient problems with cracks, *Numerical Methods for Partial Differential Equations 27 (2011),* 121–140.

[DL90] Dautray R, Lions JL. *Mathematical Analysis and Numerical Methods for Science and Technology. Integral Equations and Numerical Methods, vol. 4.* Springer: Berlin, Heidelberg, 1990.

[Mik05a] S.E. Mikhailov, Analysis of extended boundary-domain integral and integro-differential equations for a mixed BVP with variable coeffecients. Advances in Boundary Integral Methods Proccedings of the 5th UK Conference on Boundary Integral Methods, (Edited by Ke Chen),*University of Liverpool Publ., UK, ISBN 0 906370 39 6, (2005),* 106-125.

[Mik05b] S.E. Mikhailov, Localized direct boundary-domain integro-differential formulations for scalar nonlinear BVPs with variable coeffecients. *J. Eng. Math., 51 (2005),* 283-302.

[Mik06] S.E. Mikhailov, Analysis of united boundary-domain integral and integro-differential equations for a mixed BVP with variable coeffecients. *Math. Meth. Appl. Sci. 29 (2006),* 715-739.

[Mik11] S.E. Mikhailov, Traces, extensions and co-normal derivatives for elliptic systems on Lipschitz domains. *J. Math. Analysis and Appl. 378 (2011),* 324-342.