Stability Analysis of Discretized Structure Systems Based on the Complex Network with Dynamics of Time-varying Stiffness

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Abstract. The stability analysis of dynamic continuous structural system (DCSS) has often been investigated by discretizing it into several low-dimensional elements. The integrated results of all elements are employed to describe the whole dynamic behavior of DCSS. In this paper, DCSS is regarded as the complex dynamic network with the discretized elements as the dynamic nodes and the time-varying stiffness as the dynamic link relations between them, by which the DCSS can be regarded to be the large-scale system composed of the node subsystem (NS) and link subsystem (LS). Therefore, the dynamic model of DCSS is proposed as the combination of dynamic equations of NS and LS, in which their state variables are coupled mutually. By using the model, this paper investigates the stability of DCSS. The research results show that the state variables of NS and LS are uniformly ultimately bounded (UUB) associated with the synthesized coupling terms in LS. Finally, the simulation example is utilized to demonstrate the validity of method in this paper.

Keywords. Dynamic continuous structural system, Complex dynamic network, The node subsystem and link subsystem, Uniformly ultimately bounded.

1. Introduction

Complex dynamic network is regarded as the graphic model for describing the behavior of complex dynamic system, which is composed of many time-varying nodes and the relations between them. Its theoretical analysis methods are widely used in various science and industry branches such as the stabilities and associative memory in biological and artificial neural networks[1-5], the synchronization and consistency of network agents in aerospace engineering[6-10] and the vulnerabilities and stabilities of large scale electric networks[11-15]. The main purpose of using complex network methods for analyzing such systems is that the considered system consists structurally of many nodes with their link relations, and the dynamics of system can be transferred as the dynamics of nodes and their link relations, and thus the solution for concerned problem may be obtained by analyzing the dynamic behaviors of nodes and their link relations.

Inspired by the observations for the above literature, the discretizable continuous structural systems are considered in this paper. Such systems include continuous physical systems which are used in the

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structural mechanics, fluid mechanics and so on. From the perspective of research, their common characteristics are discretizable, that is, they can be partitioned theoretically into different discrete elements interconnected, and thus the dynamic analysis for whole system is transferred to analyze the dynamics of elements interconnected. Such discretization methods include mainly the lumped mass methods[16-18], the generalized coordinates methods[19-20] and the finite element methods[21-25]. In the past decades, by employing the above methods, there has been growing interest for the discretizable structural systems in displacement-based stable design because the displacements provide a more fundamental expression for the response to external forces [26-30]. In the existing literature for displacement vibration, it is seen that using the above methods cut the structure system into several elements (sub-bodies), then reconnects the adjacent elements at “nodes” satisfying some boundary conditions, and thus reduces the whole dynamic problem to the local ones.

However, it is observed that if the discrete elements are regarded as the nodes of complex network, the aforementioned methods in the above literature do not involve the complex networks, and the dynamics of link relations between nodes do not be shown. Since the whole displacement vibration of discretizable system is related directly not only to the dynamics of discrete elements but also to the coupling dynamics between them, the displacement vibration can be controlled most efficiently through the imposition of coupling dynamic links between elements rather than only the dynamics of elements. Therefore, from angle of complex dynamic network, the displacement vibration of discretizable system can be regarded to be synthesized by the two dynamics of elements and their link relations, and the two dynamics are combined together to form the whole model for analyzing the vibration. In other words, a discretizable dynamic system can be regarded as the complex dynamic network which includes the node subsystem and link subsystem which are coupled to each other.

In order to using the complex network to describing the discretizable system, it must be clear what are chosen as the link relations between network nodes. This paper focuses mainly on the displacement vibration of structural system which is discretized theoretically into several elements. It is known that most of structural systems operate under loading conditions, which may cause the displacement vibration. In this case, the stiffness of each element is relative not only to itself restoring force, but also to the restoring forces of the other elements[21-22]. Therefore, the stiffness of each element related to another element can be regarded as the link strength between the two elements. This means that the discretized structural system can be regarded as the directed weighted complex network with the elements as the nodes and their stiffness related to other elements as the link strengths. In other words, the stiffness K_{ij} is regarded as the link strength between the ith element and the jth element. Generally speaking, if each element has the n-DOF dimension, the stiffness K_{ij} is an $n\times n$ matrix, which implies that the discretized structural system can be considered as a directed multiple value-weighted network.

Analyzing a structural system according to the above method, if the stiffness is time-varying, the whole displacement vibration can be regarded as consisting of the dynamics of elements combined with the dynamics of stiffness. To the best of our knowledge, in the existing literature there is rarely research results obtained by using this method. Compared with the existing literature, the advantages of this paper are mainly stated below. **(1)** The complex network is employed to assist discretizing the structural system into several elements as the network nodes. This discretizing process is not necessary to consider the usual strict boundary conditions for assigning connective elements. The dynamic constraints between elements are reflected by the coupling relations between stiffness and elements. In other words, the time-varying stiffness influences the dynamics of elements via their mutual coupling relations and vice versa. **(2)** The dynamic equations of each element and its time-varying stiffness related to the other elements are proposed and put them together to form the vector or matrix differential equation for analyzing the whole displacement vibration. This process is not necessary to consider the integration by shape functions and piecewise polynomial interpolation. Utilizing the above advantages, this paper investigates the displacement vibrational behavior for a class of discretized dynamic structural systems with external forces. The results in this paper show that the suitable coupling relations between stiffness and elements can effectively reduce the influence of external force on displacement.

The outline of this paper is given as follows. In Section 2, utilizing the complex network method modeling a class of discretized structural systems, the dynamic equation of which is proposed with some mathematic assumptions. Section 3 discusses the stable analyzing of displacement vibration by using Lyapunov stability theory. In Section 4, the numerical example is given to show the validity of results in this paper. Conclusion are given in Section 5.

2. The Model and Assumptions

Consider a continuous physical structure system, inspired by the finite element methods, it can be discretized into N elements, where each element possesses n-DOFs. The element stiffness matrix $K_{ij} \in R^{n\times n}$ denotes the bending influence of the jth element on the ith element via the elastic restoring force. Especially, K_{ii} describes the stiffness of the ith element itself, i, j=1, 2, … , N. In this paper, K_{ij} is chosen as the link strength between the ith node and jth node. Therefore, the continuous physical structure system can be regarded as the complex network with N nodes and their link strengths K_{ij}. In this paper, the dynamic loads are assumed to act at the nodes.

Dynamics of the ith node is described by using the following differential equation.

$$M_i z_i'' + f_{di}(x') + \sum_{j=1}^{N} K_{ij} z_j = f_i(t) \tag{1}$$

where the state vectors $z_i = \left(z_{i1} \ \cdots \ z_{in} \right)^T \in R^n$, $x = \left(z_1^T \quad z_2^T \ \cdots \ z_N^T \right)^T \in R^{nN}$. $M_i \in R^{n\times n}$ is the mass matrix, $K_{ij} = K_{ij}(t) \in R^{n\times n}$ is the time-varying stiffness matrix. The damping force vector $f_{di}(x') \in R^n$, the external force vector $f_i(t) \in R^n$, i=1,2, …, N.

Remark 1. (i) In engineering applications, Equation (1) describe the forced vibration of the ith element with the displacement $z_i \in R^n$ (n-degrees of freedom). The damping force vector for the ith element $f_{di}(x')$ is relative to the whole displacement velocity vector x'. $\sum_{j=1}^{N} K_{ij} z_j$ denotes the elastic restoring force for the ith element, which is regarded as the linear interaction between the elastic restoring forces of elements. **(ii)** From the angle of complex network, the term $\sum_{j=1}^{N} K_{ij} z_j$ shows the coupling relation between nodes, in which the stiffness matrix K_{ij} denotes the multiple relationships between the ith node and the jth node.

Assumption 1. Consider the dynamic equation (1). The stiffness matrix $K_{ij} = K_{ij}(t)$ may be partitioned into $K_{ij} = K_{ij}^0 + \kappa \hat{K}_{ij}(t)$, where K_{ij}^0 is the constant stiffness matrix, $\kappa > 0$ is a real constant which denotes the common link relation strength. and $\hat{K}_{ij}(t)$ is the time-varying matrix, $i, j = 1, 2, \cdots, N$.

Remark 2. In engineering applications, Assumption 1 means that K_{ij}^0 is the static stiffness matrix, and $\hat{K}_{ij}(t)$ is regarded as the time-varying stiffness deviation matrix caused by external force or time parametric excitation. For example, K_{ij}^0 can be regarded as the stiffness of the structure when the crack is fully open in [31-32], or the matrix of average values of stiffness in [33]. $\hat{K}_{ij}(t)$ is the changing value of stiffness when the crack is closed in [31-32], or the stiffness variation matrix in [33].

Introducing the block-matrix $K = K(t) = [K_{ij}] \in R^{nN \times nN}$ denote the link relation matrix of the whole network. By using Assumption1, it is seen that $K(t) = K_0 + \kappa \hat{K}(t)$, where the constant block-matrix $K_0 = [K_{ij}^0] \in R^{nN \times nN}$, the stiffness variation block-matrix $\hat{K}(t) = [\hat{K}_{ij}(t)] \in R^{nN \times nN}$.

In this paper, $\hat{K}(t)$ is called as the stiffness variation (matrix). The dynamic equation (1) can be represented as

$$Mx'' + f_d(x') + K_0 x + \kappa \hat{K}(t) x = f(t) \tag{2}$$

where the mass matrix $M = diag\begin{pmatrix} M_1 & M_2 & \cdots & M_N \end{pmatrix}$, the external force vector $f(t) = \begin{pmatrix} f_1^T(t) & f_2^T(t) & \cdots & f_N^T(t) \end{pmatrix}^T$, the damping vector $f_d(x') = \begin{pmatrix} f_{d1}^T(x') & f_{d2}^T(x') & \cdots & f_{dN}^T(x') \end{pmatrix}^T$.

Assumption 2. The damping vector $f_d(x') = Cx'$, where $C = C_0 + \sigma G$, C_0 and G are real constant matrices, σ is a real small constant.

Remark 3. In engineering applications, Assumption 2 means that the damping force vector $f_{di}(x')$ is relative linearly with the whole displacement velocity vector x'. Generally speaking, the coefficients matrix C is called as the viscous damping matrix[34]. The decomposition form $C = C_0 + \mu G$ is also common in engineering, for example, G is called as the gyroscopic matrix in [34-39], and it is symmetric in [34-35], skew symmetric in [36-37], arbitrary square matrix in [38-39]. In this paper, G is regarded as an arbitrary square matrix similar to [38-39].

The state variables of the dynamic system (2) are chosen as $x_1 = Mx$, $x_2 = Mx'$. If Assumption 1-2 are satisfied, the dynamic system (2) can be represented as the following form.

$$\begin{cases} x_1' = x_2 \\ x_2' = -K_0 M^{-1} x_1 - CM^{-1} x_2 + f(t) - \kappa \hat{K}(t) x \end{cases} \tag{3}$$

In this paper, using the state vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in R^{2nN}$, $O_{m\times m}$ denotes the $m\times m$ zero matrix, I_m denotes the $m\times m$ identity matrix, the equation (3) can be rewritten as follows.

$$X' = AX + F(t) \tag{4}$$

where the matrix $A = \begin{bmatrix} O_{nN\times nN} & I_{nN} \\ -K_0 M^{-1} & -CM^{-1} \end{bmatrix} \in R^{2nN\times 2nN}$, the vector function $F(t) = \begin{pmatrix} O_{nN\times 1} \\ f(t) - \kappa \hat{K}(t) x \end{pmatrix}$.

Assumption 4. Consider the dynamic equations (4). A is a Hurwitz matrix.

The Assumption 4 implies that the following Lyapunov equation has the positive definition matrix solution $\Omega_1 \in R^{2nN\times 2nN}$ for the given positive definition matrices $Q_1 \in R^{2nN\times 2nN}$.

$$A^T \Omega_1 + \Omega_1 A = -Q_1 \tag{5}$$

From the angle of engineering applications, Equation (4) describes only the whole dynamics of structural system. The stiffness variation matrix $\hat{K}(t)$ is shown in the coupling term $F(t)$, which implies that the behavior of $\hat{K}(t)$ influences on the whole displacement. Therefore, an open problem is what kind of dynamic behavior of $\hat{K}(t)$ leads to the whole displacement converging asymptotically to zero. In order to obtain the solution for the problem, the dynamic mathematical model of $\hat{K}(t)$ is proposed as follows.

$$\hat{K}' = P\hat{K} + \Phi(X) \tag{6}$$

where the constant diagonal block matrix $P = diag\left(P_1 \quad P_2 \quad \cdots \quad P_N\right) \in R^{nN\times nN}$, $P_i = [p^i_{rs}] \in R^{n\times n}$, the coupled block matrix $\Phi(X) = [\Phi_{ij}(X)] \in R^{nN\times nN}$, $\Phi_{ij}(X) \in R^{n\times n}$, $i,j = 1,2,\cdots,N$.

Remark 4. (i) From the angle of engineering applications, Equation (6) shows that the stiffness variation $\hat{K}(t)$ dependents on the displacements and velocities of elements. The form of Equation (6) is inspired by the results in [40-41], in which the normalized stiffness is proposed as $k(t) = \frac{1}{0.85\varepsilon}[-\frac{1}{(\varepsilon t_z)^2}t^2 + \frac{1}{\varepsilon t_z}t + 0.55]$, where ε is a real positive constant, t_z is the meshing period. It is seen that $k(t)$ can be rewritten as $k(t) = k_0 + \mu\hat{k}(t)$, where $k_0 = \frac{2.15}{0.85\varepsilon}$, $\mu = -\frac{1.8}{0.85\varepsilon}$ and $\hat{k}(t) = 1 - \frac{1}{\varepsilon t_z}t + \frac{1}{(\varepsilon t_z)^2}t^2 \approx \exp(-\frac{1}{\varepsilon t_z}t)$. This implies that $k(t)$ approximately satisfies the differential equation $\hat{k}'(t) = -\frac{1}{\varepsilon t_z}\hat{k}(t)$. The similar result is also shown in [42]. On the other hand, the results in [42-43] show that the stiffness values between the adjacent connected elements are influenced by their displacements and velocities. Therefore, Equation (6) is proposed by combining the above two observations. **(ii)** From the perspective of complex network, Equation (6) describes the dynamics of link subsystem, in which the matrix *Φ*(*X*) represents the coupling relation between nodes and links. **(iii)** Equation (6) can be represented equivalently as follows.

$$\hat{K}'_{ij} = P_i\hat{K}_{ij} + \Phi_{ij}(X) \tag{7}$$

where the stiffness variation $\hat{K}_{ij} = (\hat{k}^{ij}_{rs}) \in R^{n\times n}$, $\Phi_{ij}(X) = [\phi^{ij}_{rs}(X)] \in R^{n\times n}$, $i,j = 1,2,\cdots,N$. Equation (7) can be rewritten equivalently as follows.

$$\frac{d\hat{k}^{ij}_{rs}}{dt} = \sum_{k=1}^{n} p^i_{rk}\hat{k}^{ij}_{ks} + \phi^{ij}_{rs}(X) \tag{8}$$

In the sense of Engineering, the time-varying stiffness value $\hat{k}^{ij}_{rs}$ in Equation (8) is explained as the force developed at the *r*th joint due to unit displacement at the *s*th joint in the *i*th element, while the *j*th element exerts a force on the *i*th element and all other elements together with all other joints in the *i*th element are fixed. Therefore, Equation (8) means that the changing rate of time-varying stiffness value $\hat{k}_{rs}^{ij}$ may be influenced directly and linearly by the other time-varying stiffness value $\hat{k}_{ks}^{ij}$ and nonlinearly by the whole network displacement and velocity vector *X*.

Assumption 5. Consider the dynamic equations (6). *P* is a Hurwitz matrix.

The Assumption 5 implies that the following Lyapunov equation has the positive definition matrix solution Ω_2 for the given positive definition matrices Q_2.

$$P^T\Omega_2 + \Omega_2 P = -Q_2 \tag{9}$$

Through the above discussion, the complex dynamic network considered in this paper is described mathematically by Equations (4) and (6) with Assumptions 1-5.

3. The Main results

Consider the complex dynamic network with (4) and (6). This paper mainly solves the problem that what conditions satisfied by the coupling term $\Phi(X)$ can make the network stable in Lyapunov sense. In order to solve the problem, introducing the symbol $vec(*)$ to denote the operator which takes the d×d matrix '*' and stacks the columns into a single vector of length d^2. It has the property $vec(MBD) = (D^T \otimes M)vec(B)$ for the matrices M, B, D with suitable dimensions[44], where '⊗' denotes the Kronecker product. Using the symbols $\hat{\xi}=\hat{\xi}(t) = vec[\hat{K}(t)] \in R^{n^2N^2}$, $\beta(x) = vec[\Phi(X)] \in R^{n^2N^2}$. It is easily seen that $vec[\hat{K}(t)x] = vec[I_{nN}\hat{K}(t)x] = [x^T \otimes I_{nN}]\hat{\xi}$.

In addition, the positive definition matrix solution $\Omega_1 \in R^{2nN\times 2nN}$ in Lyapunov equation (5) can be rewritten as the block matrix $\Omega_1 = \begin{bmatrix} \Omega_{11}^1 & \Omega_{12}^1 \\ \Omega_{21}^1 & \Omega_{22}^1 \end{bmatrix}$ with the sub-matrices $\Omega_{uv}^1 \in R^{nN\times nN}$, *u*, *v*=1,2.

By using the *vec*(*) operator, the dynamic system (6) can be rewritten as follows.

$$\hat{\xi}' = (I_{nN} \otimes P)\hat{\xi} + \beta(x) \tag{10}$$

It is noticed that if Assumption 5 is satisfied, $I_{nN} \otimes P$ is the $(nN)^2 \times (nN)^2$ Hurwitz matrix. By using Lyapunov equation (9) and $(S \otimes M)^T = S^T \otimes B^T$ and $SM \otimes BD = (S \otimes B)(M \otimes D)$ for the matrices S, M, B, D with suitable dimensions[44], it is seen that the following Lyapunov equation has the positive definition matrix solution $I_{nN} \otimes \Omega_2$ for the given positive definition matrices $I_{nN} \otimes Q_2$.

$$(I_{nN} \otimes P)^T (I_{nN} \otimes \Omega_2) + (I_{nN} \otimes \Omega_2)(I_{nN} \otimes P) = -(I_{nN} \otimes Q_2) \tag{11}$$

Now, consider the complex network with the nodes dynamics (4) and link dynamics (6). The following result is obtained.

Theorem 1. Let $\|f\| \le m_f$ with the real constant $m_f \quad 0$. If Assumption1-5 are satisfied and the coupling matrix $\Phi(X) = \kappa\varpi\Omega_2^{-1}(\Omega_{21}^1 Mx + \Omega_{22}^1 Mx')x^T + \alpha(X)$ with the adjustable parameter $\varpi > 0$ and the coupling error matrix $\alpha(X)$ satisfying $\|\alpha(X)\| \le \varepsilon \|X\|$, where the real $\varepsilon \quad 0$ satisfying that there exists a real $\gamma > 0$ such that $\varpi\lambda_{\min}(Q_1) > \varepsilon\gamma^{-1}\lambda_{\max}(\Omega_2)$ and $\lambda_{\min}(Q_2) > \varepsilon\gamma\lambda_{\max}(\Omega_2)$, then the stiffness variation $\hat{K}(t)$ is bounded and $\lim_{t \to +\infty} \hat{K}(t) = 0$, the state vector X falls asymptotically into the state domain $\{X \| \|X\| \le \mu m_f\}$, where $\mu = \dfrac{2\varpi\sqrt{\lambda_{\max}(\Omega_{12}^1\Omega_{21}^1) + \lambda_{\max}^2(\Omega_{22}^1)}}{\varpi\lambda_{\min}(Q_1) - \varepsilon\gamma^{-1}\lambda_{\max}(\Omega_2)}$, $\lambda_{\max}(*)$ and $\lambda_{\min}(*)$ denote the maximum and minimum eigenvalues of the matrix '*', respectively.

Proof. Consider the positive definition function $V(t) = \varpi X^T \Omega_1 X + \hat{\xi}^T (I_{nN} \otimes \Omega_2)\hat{\xi}$. It is seen that $\begin{pmatrix} O_{nN\times1} \\ \hat{K}(t)x \end{pmatrix}^T = \{\begin{pmatrix} O_{nN\times1} \\ vec[\hat{K}(t)x] \end{pmatrix}\}^T = \begin{pmatrix} O_{nN\times1} \\ [x^T \otimes I_{nN}]\hat{\xi} \end{pmatrix}^T = \{\begin{pmatrix} O_{nN\times n^2N^2} \\ x^T \otimes I_{nN} \end{pmatrix}\hat{\xi}\}^T = \hat{\xi}^T \begin{pmatrix} O_{nN\times n^2N^2} \\ x^T \otimes I_{nN} \end{pmatrix}^T$, then the orbit derivative of $V(t)$ along the dynamic equations (4) and (6) can be obtained as follows.

$$V'(t) = \varpi(X^T)'\Omega_1 X + \varpi X^T \Omega_1 X' + (\hat{\xi}^T)'(I_{nN} \otimes \Omega_2)\hat{\xi} + \hat{\xi}^T (I_{nN} \otimes \Omega_2)\hat{\xi}'$$

$$= \varpi[X^T A^T + F^T(t)]\Omega_1 X + \varpi X^T \Omega_1 [AX + F(t)] +$$

$$[\hat{\xi}^T(I_{nN}\otimes P^T)+\beta^T(X)]\ (I_{nN}\otimes\Omega_2)\hat{\xi}+\ \hat{\xi}^T\ (I_{nN}\otimes\Omega_2)\ [(I_{nN}\otimes P)\hat{\xi}+\beta(X)]$$

$$=\varpi X^T(A^T\Omega_1+\Omega_1 A)X+2\varpi F^T(t)\Omega_1 X+$$

$$\hat{\xi}^T[(I_{nN}\otimes P^T)(I_{nN}\otimes\Omega_2)+(I_{nN}\otimes\Omega_2)(I_{nN}\otimes P)]\ \hat{\xi}+\ 2\hat{\xi}^T(I_{nN}\otimes\Omega_2)\beta(X)$$

$$=-\varpi X^T Q_1 X+2\varpi F^T(t)\Omega_1 X\ -\hat{\xi}^T(I_{nN}\otimes Q_2)\ \hat{\xi}+\ 2\hat{\xi}^T(I_{nN}\otimes\Omega_2)\beta(X)$$

$$=-\varpi X^T Q_1 X+2\varpi\begin{pmatrix}O_{nN\times 1}\\ f-\kappa\hat{K}(t)x\end{pmatrix}^T\Omega_1 X\ -\hat{\xi}^T(I_{nN}\otimes Q_2)\ \hat{\xi}+\ 2\hat{\xi}^T(I_{nN}\otimes\Omega_2)\beta(X)$$

$$=-\varpi X^T Q_1 X+2\varpi\begin{pmatrix}O_{nN\times 1}\\ f\end{pmatrix}^T\Omega_1 X-2\kappa\varpi\begin{pmatrix}O_{nN\times 1}\\ \hat{K}(t)x\end{pmatrix}^T\Omega_1 X\ -\hat{\xi}^T(I_{nN}\otimes Q_2)\ \hat{\xi}+\ 2\hat{\xi}^T(I_{nN}\otimes\Omega_2)\beta(X)$$

$$=-\varpi X^T Q_1 X+2\varpi\begin{pmatrix}O_{nN\times 1}\\ f\end{pmatrix}^T\Omega_1 X-2\kappa\varpi\hat{\xi}^T\begin{pmatrix}O_{nN\times n^2N^2}\\ x^T\otimes I_{nN}\end{pmatrix}^T\Omega_1 X\ -\hat{\xi}^T(I_{nN}\otimes Q_2)\ \hat{\xi}+\ 2\hat{\xi}^T(I_{nN}\otimes\Omega_2)\beta(X)$$

$$=-\varpi X^T Q_1 X+2\varpi\begin{pmatrix}O_{nN\times 1}\\ f\end{pmatrix}^T\Omega_1 X\ -\hat{\xi}^T(I_{nN}\otimes Q_2)\ \hat{\xi}$$

$$-2\hat{\xi}^T\{\kappa\varpi\begin{pmatrix}O_{nN\times n^2N^2}\\ x^T\otimes I_{nN}\end{pmatrix}^T\Omega_1 X-(I_{nN}\otimes\Omega_2)\beta(X)\}\qquad(11)$$

It is seen that

$$\begin{pmatrix}O_{nN\times n^2N^2}\\ x^T\otimes I_{nN}\end{pmatrix}^T\Omega_1 X=\begin{pmatrix}O_{n^2N^2\times nN} & x\otimes I_{nN}\end{pmatrix}\Omega_1 X=\begin{pmatrix}O_{n^2N^2\times nN} & x\otimes I_{nN}\end{pmatrix}\begin{bmatrix}\Omega_{11}^1 & \Omega_{12}^1\\ \Omega_{21}^1 & \Omega_{22}^1\end{bmatrix}X$$

$$=\begin{pmatrix}[x\otimes I_{nN}]\Omega_{21}^1 & [x\otimes I_{nN}]\Omega_{22}^1\end{pmatrix}\begin{pmatrix}x_1\\ x_2\end{pmatrix}=(x\otimes I_{nN})\Omega_{21}^1 x_1+(x\otimes I_{nN})\Omega_{22}^1 x_2$$

$$=(x\otimes I_{nN})(\Omega_{21}^1 x_1+\Omega_{22}^1 x_2)\ =vec\{(\Omega_{21}^1 x_1+\Omega_{22}^1 x_2)x^T\}\qquad(12)$$

Noticing that if $\Phi(X)=\kappa\varpi\Omega_2^{-1}(\Omega_{21}^1 Mx+\Omega_{22}^1 Mx')x^T+\alpha(X)$, it has $\Omega_2\Phi(X)=\kappa\varpi(\Omega_{21}^1 x_1+\Omega_{22}^1 x_2)x^T+\Omega_2\alpha(X)$, and thus it has $vec[\Omega_2\Phi(X)]=\kappa\varpi vec\{(\Omega_{21}^1 x_1+\Omega_{22}^1 x_2)x^T\}+vec[\Omega_2\alpha(X)]$, by using $vec[\Omega_2\Phi(X)]=(I_{nN}\otimes\Omega_2)\beta(X)$ and (12), it has

$(I_{nN}\otimes\Omega_2)\beta(X)=\kappa\varpi\begin{pmatrix}O_{nN\times n^2N^2}\\ x^T\otimes I_{nN}\end{pmatrix}^T\Omega_1 X+\ vec[\Omega_2\alpha(X)]$. Substituting this result into the equation (11), it is obtained that

$$V'(t) = -\varpi X^T Q_1 X + 2\varpi \begin{pmatrix} O_{nN\times 1} \\ f \end{pmatrix}^T \Omega_1 X \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\hat{\xi}^T vec[\Omega_2 \alpha(X)]$$

$$= -\varpi X^T Q_1 X + 2\varpi \begin{pmatrix} O_{1\times nN} & f^T \end{pmatrix} \Omega_1 X \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\hat{\xi}^T (I_{nN} \otimes \Omega_2) vec[\alpha(X)]$$

$$= -\varpi X^T Q_1 X + 2\varpi \begin{pmatrix} O_{1\times nN} & f^T \end{pmatrix} \begin{bmatrix} \Omega_{11}^1 & \Omega_{12}^1 \\ \Omega_{21}^1 & \Omega_{22}^1 \end{bmatrix} X \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\hat{\xi}^T (I_{nN} \otimes \Omega_2) vec[\alpha(X)]$$

$$= -\varpi X^T Q_1 X + 2\varpi \begin{pmatrix} f^T \Omega_{21}^1 & f^T \Omega_{22}^1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\hat{\xi}^T (I_{nN} \otimes \Omega_2) vec[\alpha(X)]$$

$$= -\varpi X^T Q_1 X + 2\varpi (f^T \Omega_{21}^1 x_1 + f^T \Omega_{22}^1 x_2) \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\hat{\xi}^T (I_{nN} \otimes \Omega_2) vec[\alpha(X)]$$

$$= -\varpi X^T Q_1 X + 2\varpi f^T (\Omega_{21}^1 x_1 + \Omega_{22}^1 x_2) \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\hat{\xi}^T (I_{nN} \otimes \Omega_2) vec[\alpha(X)]$$

$$\leq -\varpi X^T Q_1 X + 2\varpi \|f\| \cdot \|\Omega_{21}^1 x_1 + \Omega_{22}^1 x_2\| \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\|\hat{\xi}\| \cdot \|(I_{nN} \otimes \Omega_2) vec[\alpha(X)]\|$$

$$\leq -\varpi X^T Q_1 X + 2\varpi \|f\| \cdot (\|\Omega_{21}^1 x_1\| + \|\Omega_{22}^1 x_2\|) \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\|\hat{\xi}\| \cdot \sqrt{\|(I_{nN} \otimes \Omega_2) vec[\alpha(X)]\|^2}$$

$$\leq -\varpi X^T Q_1 X + 2\varpi m_f (\sqrt{\|\Omega_{21}^1 x_1\|^2} + \sqrt{\|\Omega_{22}^1 x_2\|^2}) \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\lambda_{\max}(\Omega_2) \|\hat{\xi}\| \cdot \|vec[\alpha(X)]\|$$

$$= -\varpi X^T Q_1 X + 2\varpi m_f (\sqrt{x_1^T \Omega_{12}^1 \Omega_{21}^1 x_1} + \sqrt{x_2^T (\Omega_{22}^1)^2 x_2}) \ -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\lambda_{\max}(\Omega_2) \|\hat{\xi}\| \cdot \|\alpha(X)\|$$

$$\leq -\varpi X^T Q_1 X + 2\varpi m_f (\sqrt{\lambda_{\max}(\Omega_{12}^1 \Omega_{21}^1)}\, \|x_1\| + \lambda_{\max}(\Omega_{22}^1) \|x_2\|) -\hat{\xi}^T (I_{nN} \otimes Q_2)\ \hat{\xi} \ +2\varepsilon\lambda_{\max}(\Omega_2) \|\hat{\xi}\| \cdot \|X\|$$

$$\leq -\varpi \lambda_{\min}(Q_1) \|X\|^2 + 2\varpi m_f \sqrt{\lambda_{\max}(\Omega_{12}^1 \Omega_{21}^1) + \lambda_{\max}^2(\Omega_{22}^1)} \cdot \sqrt{\|x_1\|^2 + \|x_2\|^2}$$

$$-\lambda_{\min}(Q_2) \|\hat{\xi}\|^2 + \varepsilon \lambda_{\max}(\Omega_2)(\gamma \|\hat{\xi}\|^2 + \gamma^{-1} \|X\|^2)$$

$$= -[\varpi \lambda_{\min}(Q_1) - \varepsilon \gamma^{-1} \lambda_{\max}(\Omega_2)] \|X\|^2 + 2\varpi m_f \sqrt{\lambda_{\max}(\Omega_{12}^1 \Omega_{21}^1) + \lambda_{\max}^2(\Omega_{22}^1)}\, \|X\|$$

$$-[\lambda_{\min}(Q_2) - \varepsilon \gamma \lambda_{\max}(\Omega_2)] \|\hat{\xi}\|^2$$

$$= -[\varpi \lambda_{\min}(Q_1) - \varepsilon \gamma^{-1} \lambda_{\max}(\Omega_2)](\|X\| - \mu m_f) \|X\| \ -[\lambda_{\min}(Q_2) - \varepsilon \gamma \lambda_{\max}(\Omega_2)] \|\hat{\xi}\|^2 \tag{13}$$

From (13), it is seen that if the conditions in Theorem 1 are satisfied and $\|X\| > \mu m_f$, then $V'(t) \leq 0$, by using the results in [45], the proof of Theorem 1 is completed.

Observing the proof of Theorem 1, the following result is obvious.

Corollary 1. If $f(t)=0$, Assumption1-5 and the other conditions in Theorem 1 are satisfied, then the complex dynamic network with (4) and (6) is asymptotically stable in Lyapunov sense. This means that $\lim_{t\to+\infty} X(t)=0$ and $\lim_{t\to+\infty} \hat{K}(t)=0$, which implies that $\lim_{t\to+\infty} x(t)=0, \lim_{t\to+\infty} x'(t)=0$ by using $x(t)=M^{-1}x_1$, $x'(t)=M^{-1}x_2$.

Remark 5. For the purpose of applications, Theorem 1 can be further explained as follows.

(i). The inequality $\|X\|\le \mu m_f$ implies that $\|x\|\le \dfrac{\mu m_f}{\lambda_{\min}(M)}$ and $\|x'\|\le \dfrac{\mu m_f}{\lambda_{\min}(M)}$ by using the result $\|X\|=\sqrt{x^T M^2 x + x'^T M^2 x'} \quad \lambda_{\min}(M)\sqrt{\|x\|^2+\|x'\|^2}$. This shows that the minimum eigenvalue of mass matrix M influences the size of whole displacement and velocity. Noticing $M=diag\left(M_1 \quad M_2 \quad \cdots \quad M_N\right)$, the above result is equivalent to saying that the minimum one of eigenvalues in all element mass matrices M_i directly affects the size of whole displacement and velocity.

(ii). If the parameter ε=0 (this means $\Phi(X)=\kappa\varpi\Omega_2^{-1}(\Omega_{21}^1 Mx+\Omega_{22}^1 Mx')x^T$), the inequalities $\varpi\lambda_{\min}(Q_1)>\varepsilon\gamma^{-1}\lambda_{\max}(\Omega_2)$ and $\lambda_{\min}(Q_2)>\varepsilon\gamma\lambda_{\max}(\Omega_2)$ are satisfied for any real parameter γ>0, or if $\varepsilon\neq0$, γ is chosen as $\dfrac{\varepsilon\lambda_{\max}(\Omega_2)}{\varpi\lambda_{\min}(Q_1)}<\gamma<\dfrac{\lambda_{\min}(Q_2)}{\varepsilon\lambda_{\max}(\Omega_2)}$ with Q_1 chosen as $Q_1-\dfrac{\varepsilon^2\lambda_{\max}^2(\Omega_2)}{\varpi\lambda_{\min}(Q_2)}I_{2nN}>0$, which is equivalent to $\dfrac{\varepsilon\lambda_{\max}(\Omega_2)}{\lambda_{\min}(Q_2)}<\gamma^{-1}<\dfrac{\varpi\lambda_{\min}(Q_1)}{\varepsilon\lambda_{\max}(\Omega_2)}$ with the same Q_1.

(iii). From (ii), it is noticing that if $\varepsilon\neq0$ and choosing $\gamma=(1-\theta_1)\dfrac{\lambda_{\max}(\Omega_2)}{\varpi\lambda_{\min}(Q_1)}\varepsilon+\theta_1\dfrac{\lambda_{\min}(Q_2)}{\varepsilon\lambda_{\max}(\Omega_2)}$ or $\gamma^{-1}=(1-\theta_2)\dfrac{\varepsilon\lambda_{\max}(\Omega_2)}{\lambda_{\min}(Q_2)}+\theta_2\dfrac{\varpi\lambda_{\min}(Q_1)}{\varepsilon\lambda_{\max}(\Omega_2)}$, and $Q_1=[\dfrac{\varepsilon^2\lambda_{\max}^2(\Omega_2)}{\varpi\lambda_{\min}(Q_2)}+\gamma_0]I_{2nN}$ with the arbitrary real constants $0<\theta_1<1$, $0<\theta_2<1$, $\gamma_0>0$, then it is seen that $\varpi\lambda_{\min}(Q_1)-\varepsilon\gamma^{-1}\lambda_{\max}(\Omega_2)=(1-\theta_2)\varpi\gamma_0$ and $\lambda_{\min}(Q_2)-\varepsilon\gamma\lambda_{\max}(\Omega_2)=\dfrac{(1-\theta_1)\varpi\gamma_0\lambda_{\min}^2(Q_2)}{\varepsilon^2\lambda_{\max}^2(\Omega_2)+\varpi\gamma_0\lambda_{\min}(Q_2)}$, the inequalities $\varpi\lambda_{\min}(Q_1)>\varepsilon\gamma^{-1}\lambda_{\max}(\Omega_2)$ and

$\lambda_{\min}(Q_2) > \varepsilon\gamma\lambda_{\max}(\Omega_2)$ are satisfied. Therefore, it is obtained that

$$\mu = \frac{2\sqrt{\lambda_{\max}(\Omega_{12}^{1}\Omega_{21}^{1}) + \lambda_{\max}^{2}(\Omega_{22}^{1})}}{(1-\theta_2)\gamma_0} \tag{14}$$

By using (14), it is seen that the following result is true.

Corollary 2. Let $\delta > 0$ be a given small positive real, and the parameters and Q_1 is chosen by using (iii) in Remark 5. If Assumption1-5 are satisfied, $\|f\| \le m_f$ (m_f 0) with $m_f \le \frac{\delta\gamma_0}{2\sqrt{\lambda_{\max}(\Omega_{12}^{1}\Omega_{21}^{1}) + \lambda_{\max}^{2}(\Omega_{22}^{1})}}$, then $\mu m_f \le \delta(1-\theta_2)^{-1}$.

Remark 6. Corollary 2 shows in the sense of Engineering that for the forced vibration ($f(t) \ne 0$), with the help of the suitable coupling dynamics of the stiffness variation $\hat{K}(t)$ and the above chosen parameters, the sizes of displacement and velocity of each element is approximation to the given small positive real δ. Especially, the suitable coupling dynamics of $\hat{K}(t)$ can make the displacement and velocity of each element converge asymptotically to zero in the case of unforced vibration ($f(t) = 0$), which means that the system described as the complex network with (4) and (6) is wholly asymptotically stable in Lyapunov sense.

Remark 7. The damping matrix $C = C_0 + \sigma G$ and the static stiffness matrices K_0 play the important role in the result of Theorem 1 due to Lyapunov equation (5) with the Hurwitz matrix $A = \begin{bmatrix} O_{nN\times nN} & I_{nN} \\ -K_0M^{-1} & -CM^{-1} \end{bmatrix}$. If the real σ, matrices M and C_0 are known, the static stiffness matrices K_0 and G can be obtained by using the following algorithm steps.

Step 1. The matrix *A* can be rewritten as $A = \overline{A} + \overline{B}\overline{K}$, where $\overline{A} = \begin{bmatrix} O_{nN\times nN} & I_{nN} \\ O_{nN\times nN} & -C_0M^{-1} \end{bmatrix}$, $\overline{B} = \begin{bmatrix} O_{nN\times nN} \\ -I_{nN} \end{bmatrix}$ and $\overline{K} = \begin{pmatrix} K_0M^{-1} & \sigma GM^{-1} \end{pmatrix}$.

Step 2. Substituting $A = \overline{A} + \overline{B}\overline{K}$ into Lyapunov equation (5) with the matrix Q_1 chosen by Remark 5, construct the linear matrix inequality (LMI) $\overline{A}\Omega_1 + \Omega_1\overline{A}^T + \overline{B}Y + Y^T\overline{B}^T + Q_1 < 0$.

Step 3. By using the LMI methods in [46-47], the LMI solution matrices Ω_1 and Y in Step 2 are obtained, by which the matrices K_0 and G are obtained by using $\bar{K} = \left(K_0 M^{-1} \quad \sigma G M^{-1} \right) = Y \Omega_1^{-1}$.

4. Simulation

Consider the transverse plane vibration of elastic beam with non-uniform mass characteristics. Let us use lumped mass elements to model the elastic beam. Such lumped models can be easily regarded as the complex dynamic network in which the stiffness matrices are chosen as the relations between lumped mass elements.

The elastic beam is discretized as N elements which's serial number 1, 2, … , N (Fig.1). If N is big enough, each element can be regarded as the approximation uniform segment which can be lumped on the two block-mass. The displacement of the two block-mass represents the displacement of element. For example, $z_i = \left(z_{i1} \quad z_{i2} \right)^T$ and $M_i = diag\left(m_{i1} \quad m_{i2} \right)$ in Fig.1 denote the displacement and mass matrix of the ith element with the external force $f_i(t)$, respectively, i=1,2,…, N.

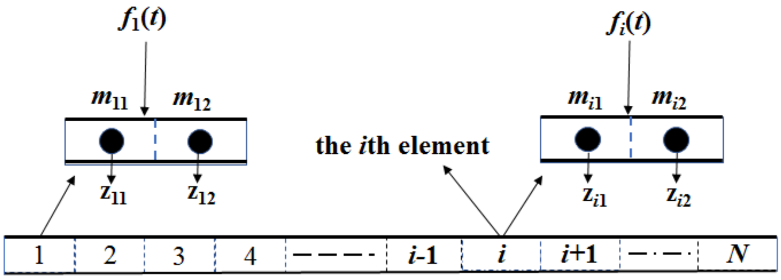

Fig.1 The diagram of discretized elastic beam with non-uniform characteristics

It is noticing that the elastic restoring force of each element is relative not only to its own stiffness but also to the stiffness of other elements, and thus by using D'alembert principle for each element, the governing equation of each element with the damping force can be expressed in the form of (1).

In the following simulation by Matlab, the length of elastic beam is L, the state equation of the ith element is chosen as (1) and the state equation of stiffness variation $\hat{K}(t)$ is chosen as (6). The other parameters and matrices are chosen according to the following steps.

Step 1. The mass matrix $M_i = diag\left(m_{i1} \quad m_{i2} \right)$ of the ith element is generated with the $m_{i1} = \varepsilon_i \rho_{i1} l_i$

and $m_{i2} = (1-\varepsilon_i)\rho_{i2} l_i$, where the adjustable parameter $0 < \varepsilon_i < 1$, ρ_{i1}, ρ_{i2} are the mass densities of the two segments of the *i*th element, respectively, l_i is the length of the *i*th element satisfying $\sum_{i=1}^{N} l_i = L$. ε_i is chosen arbitrarily as $rand(1)$, ρ_{i1}, ρ_{i2} are chosen arbitrarily by $2*rand(1)$, $3*rand(1)$, respectively. l_i is chosen as $l_i = \dfrac{L}{N}$ for *i*=1, 2, …, *N*. The damping block-matrix C_0 is chosen by choosing arbitrarily as $10*rand(2*N)$. The parameter σ is chosen arbitrarily as $rand(1)$.

Step 2. Let $\Lambda_k = diag\begin{pmatrix} a_k & b_k \end{pmatrix}$, *k*=1, 2, …, *N*, be *N* diagonal matrices with negative reals a_k and b_k chosen arbitrarily by $-rand(1)$, respectively, and generating *N* invertible 2×2 matrices W_k chosen arbitrarily by $rand(2)$, respectively. Let $P_k = W_k^{-1}\Lambda_k W_k$ and $P = diag\begin{pmatrix} P_1 & P_2 & \cdots & P_N \end{pmatrix}$. This means that *P* is a Hurwitz matrix.

Step 3. Solving Lyapunov equation (9) for $Q_2 = \tau I_{2N}$ with the parameter τ chosen arbitrarily as $rand(1)$, then the positive definition Ω_2 is obtained.

Step 4. Let the external force $f_k(t) = \begin{pmatrix} m_{k1}^f \sin(\omega_k t) & m_{k2}^f \cos(\omega_k t) \end{pmatrix}^T$, *k*=1, 2, …, *N*, where the amplitudes m_{k1}^f and m_{k2}^f are chosen arbitrarily by $rand(1)$, respectively. The angular frequencies ω_k are chosen arbitrarily by $5*rand(1)$, respectively. Let $m_f = \sqrt{\sum_{k=1}^{N}[(m_{k1}^f)^2 + (m_{k2}^f)^2]}$.

Step 5. The coupling error matrix in (6) $\alpha(X) = [\alpha_{ij}(X)]$ with $\alpha_{ij}(X) = U_i\begin{bmatrix} z_i & z_i' \end{bmatrix} + V_j\begin{bmatrix} z_j & z_j' \end{bmatrix}$, where the constant matrices $U_i, V_j \in R^{2\times 2}$, *i, j*=1, 2, …, *N*, are chosen arbitrarily by $randn(2)$, respectively. The common stiffness strength κ =0.8, the parameter ϖ =0.001.

Step 6. The given positive real δ is chosen arbitrarily as $0.1*rand(1)$. Solving LMI $\bar{A}\Omega_1 + \Omega_1\bar{A}^T + \bar{B}Y + Y^T\bar{B}^T + Q_1 < 0$ for $Q_1 = [\dfrac{\varepsilon^2\lambda_{\max}^2(\Omega_2)}{\lambda_{\min}(Q_2)} + \gamma_0]I_{2nN}$ and $\gamma_0 = 2\delta^{-1}m_f\nu$, where ε is chosen arbitrarily as $10*rand(1)$ and ν is chosen arbitrarily as $2000*rand(1)$. The solution matrices

Ω_1 and Y are obtained, by which the stiffness matrix K_0 and damping matrix G are obtained by using $\overline{K} = \left(K_0 M^{-1} \quad \sigma G M^{-1} \right) = Y \Omega_1^{-1}$, respectively.

By using Step 1-6 with L=5, N=10, the initial state $x(0)$, $x'(0)$ and $vec[\hat{K}(0)]$ are chosen as *rand*(n*N,1), 10**rand*(n*N,1) and 10**rand*((n*N)^2,1), respectively. Noticing that the displacement $x(t) = M^{-1} x_1(t)$, the displacement velocities $x'(t) = M^{-1} x_2(t)$, the simulation results are in Fig.2-Fig.4.

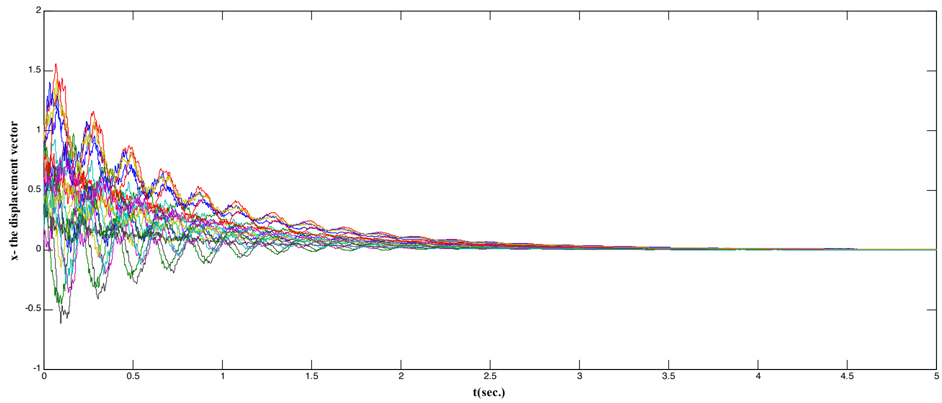

Fig.2 The time response of displacement $x(t)$ for all nodes

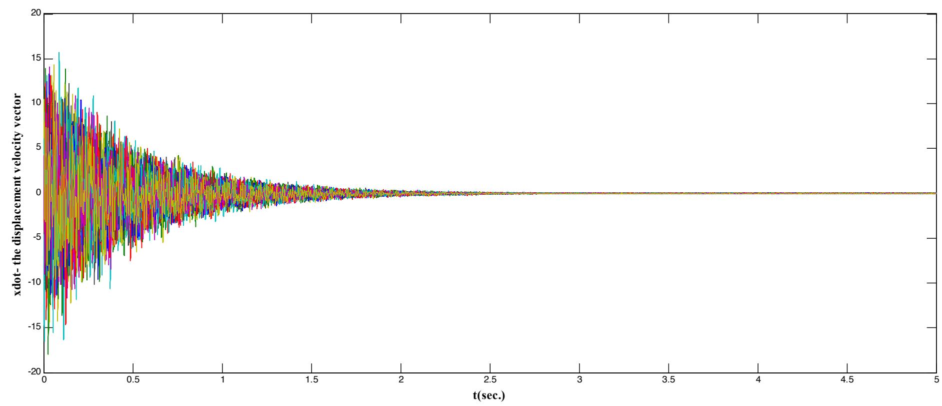

Fig.3 The time response of displacement velocity $x'(t)$ for all nodes

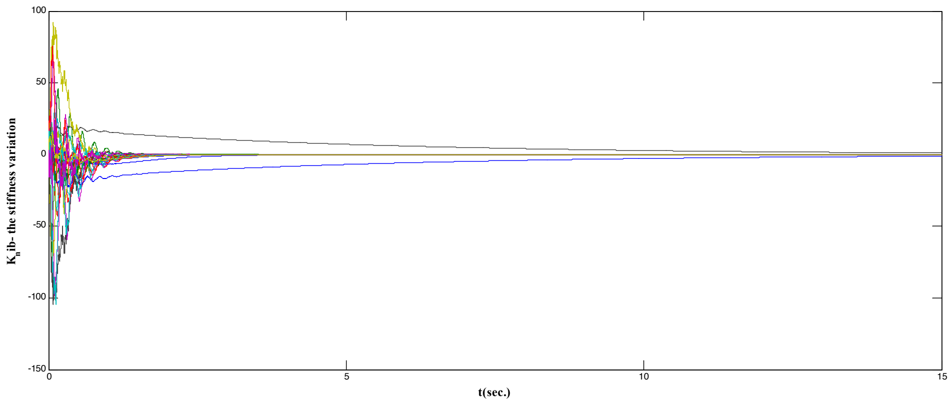

Fig.4 The time response of stiffness variation $\hat{K}(t)$ for all nodes

From Fig.2-Fig.4, the following observing results can be obtained.

(i). With the dynamics of stiffness variation $\hat{K}(t)$, the displacement vector $x(t)$ and velocity vector $x'(t)$ converge respectively to zero rapidly when the external force periodically acts on the beam, but the convergence of the stiffness variation $\hat{K}(t)$ to zero is slower (shown in Fig.4), which shows the persistence of stiffness against the deformation caused by external dynamic loads.

(ii). It is observed by Fig.2-Fig.4 that although the external force is periodic, the displacement $x(t)$ and velocity $x'(t)$ decrease aperiodically, which may be caused by the asymptotically stable linear part in the displacement motion equation (4) with the suitable damping matrix $C = C_0 + \sigma G$ and the static stiffness matrices K_0.

(iii). The parameters κ and ϖ are shown in the coupling matrix $\Phi(X)$ in (6), and their sizes not only directly affect the dynamics of stiffness variation $\hat{K}(t)$, but also affect the displacement $x(t)$ and velocity $x'(t)$ via the coupling term $f(t) - \kappa\hat{K}(t)x$ in Equation (4).

5. Conclusion

This paper considers the dynamic continuous structural system (DCSS) which can be discretized into several low-dimensional elements. All elements and the stiffness between them are integrated as the complex dynamic network, in which the element is regarded as the node and the stiffness between elements are regarded as the link relations. If the elements and stiffness are time-varying, DCSS can be regarded as consisting of the node subsystem and link subsystem, and the two subsystems are coupled to each other. The displacement vibration is described by the node subsystem, and the time-varying stiffness variation is described by the link subsystem. In this paper, the stiffness variation is regarded as an important quantity that influences the displacement vibration of DCSS via the coupled parts between the two subsystems. According to the Lyapunov stability theory, it has been proved that the vibration UUB stability of DCSS can be ensured by synthesizing the suitable coupled parts when the external force acts on the system. Since the relationship between stiffness variation and displacement deformation is not very clear, the more work in the future is required to precisely modeling the link subsystem.

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