Exponential stability of linear systems under a class of Desch-Schappacher perturbations

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Abstract

In this paper we investigate the uniform exponential stability of the system $\frac{dx(t)}{dt} = Ax(t) - \rho Bx(t)$, $(\rho > 0)$, *where the unbounded operator* A *is the infinitesimal generator of a linear* C_0*−semigroup of contractions* $S(t)$ *in a Hilbert space* X *and* B *is a Desch-Schappacher operator. Then we give sufficient conditions for exponential stability of the above system. The obtained stability result is then applied to show the uniform exponential stabilization of bilinear partial differential equations.*

Keywords: Exponential stabilization, linear system, bilinear control, unbounded control operator.

I. Introduction

Consider the following abstract system

$$\begin{cases} \dot{x}(t) = Ax(t) - \rho Bx(t), \quad t > 0 \\ x(0) = x_0, \end{cases} \tag{1}$$

where the state $x(.)$ takes values in a Hilbert state space X endowed with an inner product $\langle ., . \rangle_X$ with associate norm $\|.\|_X$, the unbounded operator $(A, D(A))$ generates a C_0-semigroup $S(t)$ on X. Here, B is an unbounded linear operator of X in the sense that it is bounded from X to some extrapolating space of X. In the case of various real problems, the modeling may lead to mathematical model of the form (1) with an operator B which is of type Desch-Schappacher. Such a perturbation operator B appears for instance in case of control actions exercised through the boundary of the geometrical domain of partial differential equations, and also in many other situations of internal control. The problem of stability of system like (1) can be viewed as a problem of unbounded perturbations of the generator domain. Indeed, to make clear this connection, define an operator $A_M = A_m$ with $D(A_M) = \{x \in D(A_m) | Gx = Mx\}$, for some boundary operator $G : Z \to X$ and bounded operator $M : X \to X$, and where $A_m : Z \to X$ is a differential operator such that $A := A_m$ with domain $D(A) = KerG$. We see then that M is a perturbation of the domain of A (see [4, 7, 8] and the references therein).

The solution of (1) does not exist, in general, with values in X. Thus, to confront this difficulty the concept of admissibility is needed, which requires the introduction of interpolating and extrapolating spaces of the state space X.

Our goal in this paper is to investigate the uniform exponential stability of the system (1). This consists on looking for a set of parameters ρ for which there exists a global $X-$valued mild solution $x(t)$ of (1) and is such that $\|x(t)\| \leq Ke^{-\sigma t}\|x_0\|,\ \forall t \geq 0$ for some constants $K, \sigma > 0$. As an application, one can consider the stabilization of bilinear systems by means of switching controllers, which leads to a closed-loop system like (1). This problem has been considered in [11] for a bounded operator B. The case of a Miyadera-Voigt type operator has been investigated in [15]. Moreover, in [2] the case of $1-$admissibility in Banach space has been considered. However, the $1-$admissibility assumption prevents us to consider the case of Hilbert state space as in this case the operator B will be necessary bounded (see [18]). In other words, the $1-$admissibility condition excludes several applications that are also available in Hilbert space. Moreover, in [2] it was assumed that $D\big((A_{-1} - \rho B)|X\big) = D(A_{-1}) \cap D(B|X)$, which played an essential role in the proofs of the stabilization results (in a technical point of view). Unfortunately, there are several examples in which this domain condition is not fulfilled (see e.g Examples 1&2). In this paper, we will rather use the $p-$admissibility property with $p \geq 1$. Then we introduce new sufficient conditions for uniform exponential stability of system (1), which are easily checkable. In the sequel, we proceed as follows: The main results of this paper are contained in Section 2. In Section 3, we provide applications to feedback stabilization of bilinear heat and transport equations.

II. Exponential stability

In this section, we state and prove our two main stabilization results. We start by introducing the necessary tools regarding the notion of admissibility in connection with the generation results and then provide some a priori estimations of the solution.

i. Preliminary results

As pointed out in the introduction, the unbounded aspect of the operator B do not guarantee the existence of an $X-$valued solution $x(t)$ of (1). However, one may extend the system at hand in a larger (extrapolating) space X_{-1} of the state space X in which the existence of the solution $x(t)$ is ensured and then give the required admissibility conditions of B, so that the solution $x(t)$ lies in X. Classically, the space X_{-1} can be viewed as the completion of X with respect to the norm $\|x\|_{-1} := \|(\lambda I - A)^{-1}x\|_X$, $x \in X$, for some λ in the resolvent set $\rho(A)$ of A. This space is independent of the choice of λ and we have the following continuous and dense embedding: $X \hookrightarrow X_{-1}$. Moreover, X_{-1} is the dual of $D(A^*)$ with respect to the pivot space X. That way the unbounded operator B becomes bounded from X to the extrapolating space X_{-1}, i.e, $B \in \mathcal{L}(X, X_{-1})$. Thus, in order to give a meaning to solutions of (1), we have to use the fact that the semigroup $S(t)$ can be extended to a C_0-semigroup $S_{-1}(t)$ on X_{-1}, whose generator A_{-1} has $D(A_{-1}) = X$ as domain and is such that $A_{-1}x = Ax$, for any $x \in D(A)$. Recall that for any given initial state $x_0 \in X$, a mild solution of (1) is an $X-$valued continuous function x on $[0, T]$ satisfying the following variation of parameters formula:

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \ \ \forall t \geq 0,$$

which always makes sense in X_{-1}. The system (1) can be rewritten in the large space X_{-1} in the following abstract form:

$$\begin{cases} \dot{x}(t) = A_{-1}x(t) - \rho Bx(t), \\ x(0) = x_0. \end{cases} \tag{2}$$

which is well-posed in X whenever $A-\rho B$ is the generator of a C_0-semigroup on X (cf. [6], Section II.6). The well-posedeness of systems like (1) has been studied in many works using different approaches (see e.g. [1, 3, 5, 9, 13, 18]).

The next result provides sufficient conditions on a Desch-Schappacher perturbation B to guarantee the existence and uniqueness of the mild solution of (1) (see [1] & ([6], p. 183)).

Theorem II.1 *Let A be the generator of a C_0-semigroup $S(t)$ on X and let $B \in \mathcal{L}(X, X_{-1})$ be $p-$admissible for some $1 \leq p < \infty$, i.e., there is a $T > 0$ such that*

$$\int_0^T S_{-1}(T-t)Bu(s)ds \in X, \quad \forall u \in L^p(0,T;X). \tag{3}$$

Then for any ρ, the operator $(A_{-1}-\rho B)|_X$ defined on the domain $D((A_{-1}-\rho B)|_X) := \{x \in X : (A_{-1}-\rho B)x \in X\}$ by

$$(A_{-1}-\rho B)|_X x := A_{-1}x - \rho Bx, \ \ \forall x \in D((A_{-1}-\rho B)|_X) \tag{4}$$

is the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on X, which verifies the variation of parameters formula

$$T(t)x = S(t)x - \rho \int_0^t S_{-1}(t-s)BT(s)xds, \ \ \forall t \geq 0, \ \forall x \in D((A_{-1}-\rho B)|_X).$$

An operator $B \in \mathcal{L}(X, X_{-1})$ satisfying the condition (3) is called a Desch-Schappacher operator or perturbation. Moreover, the operator defined by (4) is the part $(A_{-1}-\rho B)|_X$ of $(A_{-1}-\rho B)$ on X $\big($see ([16], p. 39) and ([6], p. 147)$\big)$.

Remark 1 *Notice that since $W^{1,p}(0,T;X)$ is dense in $L^p(0,T;X)$, the range condition (3) is equivalent to the existence of some $M > 0$ such that*

$$\left\| \int_0^T S_{-1}(T-s)Bu(s)ds \right\|_X \leq M\|u\|_{L^p(0,T;U)}, \ \ \forall u \in W^{1,p}(0,T;X), \tag{5}$$

with $\|u\|_{L^p(0,T;X)} = \left(\int_0^T \|u(t)\|_X^p dt \right)^{\frac{1}{p}}$.

Remark 2 *Note that if the operator $B \in \mathcal{L}(X, X_{-1})$ is $p-$admissible in $[0,T]$, then it is so in $[0,t]$ for any $t \in [0,T]$ (see [18]). In other words, if (5) holds then for all $t \in [0,T]$ we have the following inequality*

$$\left\| \int_0^t S_{-1}(t-r)Bu(r)dr \right\|_X \leq M\|u\|_{L^p(0,t;X)}. \tag{6}$$

Let us now show the following lemma that will be needed in the sequel.

Lemma II.2 *Let A be the generator of C_0-semigroup of contractions $S(t)$ on X and let $B \in \mathcal{L}(X, X_{-1})$ be $p-$admissible for some $1 \leq p < \infty$. Then for any $0 < \rho < \frac{1}{T^{\frac{1}{p}}M}$, the mild solution $x(t)$ of the system (1) satisfies the following estimate*

$$\|x(.)\|_{L^p(0,T;X)} \leq \frac{T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X, \ \forall x_0 \in X \tag{7}$$

and

$$\left\| \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X \leq M_\rho \|x_0\|_X, \ \ \forall t \in [T, 2T], \ \forall x_0 \in X,$$

with $M_\rho := \frac{MT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}} M} \left(2 + \rho M T^{\frac{1}{p}} \right)$.

Proof 1 *Let $x_0 \in D((A_{-1} - \rho B)|_X)$. From Theorem II.1, we know that the system (1) admits a unique mild solution $x(t)$ which is given by*

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \ \ \forall t \geq 0. \tag{8}$$

Let us estimate $\|x(\cdot)\|_{L^p(0,T;X)}$. From (8), we get via Minkowski's inequality

$$\|x(.)\|_{L^p(0,T;X)} \leq \left(\int_0^T \|S(t)x_0\|_X^p dt \right)^{\frac{1}{p}} + \rho \left(\int_0^T \left\| \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X^p \right)^{\frac{1}{p}} dt.$$

Then from Remark 2 we derive

$$\|x(\cdot)\|_{L^p(0,T;X)} \ \ \leq \ \ T^{\frac{1}{p}} \|x_0\|_X + \rho T^{\frac{1}{p}} M \|x(.)\|_p,$$

where $\|x(\cdot)\|_{L^p(0,T;X)} := \left(\int_0^T \|x(\tau)\|_X^p d\tau \right)^{\frac{1}{p}}$*, which gives the estimate (7) for any* $0 < \rho < \frac{1}{T^{\frac{1}{p}} M}$.
Now, since the mapping $x_0 \mapsto x(t)$ defines a C_0-semigroup $T(t)$ on X, the mapping $x_0 \mapsto x(\cdot) = T(\cdot)x_0$ is continuous from X to $L^p(0,T;X)$. Then the estimate (7) holds by density for any $x_0 \in X$. Let $x_0 \in X$, and let us write for any $t \in [T, 2T]$,

$$\begin{aligned} \int_0^t S_{-1}(t-s)Bx(s)ds \ \ &= \ \ \int_0^T S_{-1}(t-s)Bx(s)ds + \int_T^t S_{-1}(t-s)Bx(s)ds \\ &:= \ \ L_1 + L_2 \end{aligned}$$

Then we consider the two terms of the sum separately. For the first one, the admissibility of B together with the contraction property of $S_{-1}(t)$ yields

$$\|L_1\|_X \ \ = \ \ \left\| S_{-1}(t-T) \int_0^T S_{-1}(T-s)Bx(s)ds \right\|_X \leq M \|x(\cdot)\|_{L^p(0,T;X)} \tag{9}$$

For the second term, observing that $L_2 = \int_0^{t-T} S_{-1}(t-T-\tau)Bx(\tau+T)d\tau$, we obtain again from the admissibility of B

$$\|L_2\|_X \leq M \|x(.+T)\|_{L^p(0,T;X)}.$$

Based on the V.C.F (8), it follows directly from Lemma (7) that for all $t \geq 0$, we have

$$\|x(t)\|_X \leq \left(1 + \frac{\rho M T^{\frac{1}{p}}}{1 - T^{\frac{1}{p}} \rho M} \right) \|x_0\|_X, \forall x_0 \in X \tag{10}$$

for any $0 < \rho < \frac{1}{T^{\frac{1}{p}} M}$*. Using the last estimate, we derive the following inequalities:*

$$\|x(.+T)\|_{L^p(0,T;X)} \leq T^{\frac{1}{p}} \left(1 + \frac{\rho M^2 T^{\frac{1}{p}}}{1 - \rho M T^{\frac{1}{p}}} \right) \|x_0\|_X. \tag{11}$$

Combining (9) and (11), we obtain the desired estimate:

$$\left\| \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X \leq M_\rho \|x_0\|_X.$$

ii. A direct approach

Theorem II.3 *Let $B \in \mathcal{L}(X, X_{-1})$ and let A be the infinitesimal generator of a linear C_0-semigroup of contractions $S(t)$ on X, and assume that for some $T > 0$, we have*

(i) there exists $1 < p < \infty$ such that for all $u \in L^p(0, T; X)$, we have $\int_0^T S_{-1}(T-s)Bu(s)ds \in X$,

(ii) for some $\delta > 0$ we have

$$\int_0^T Re \left\langle S(t)x, B^*S(t)x \right\rangle_X dt \geq \delta \|S(T)x\|_X^2, \quad \forall x \in X. \tag{12}$$

Then there is a $\rho_1 > 0$ such that for all $\rho \in (0, \rho_1)$, the system (1) is exponentially stable on X.

Proof 2 *Foe any $\rho > 0$, we set $A_{\rho B} := (A_{-1} - \rho B)|_X$. According to assumption (i), we deduce from Theorem 1 that the system (1) admits a unique mild solution which is given, for $x_0 \in D(A_{\rho B})$, by the variation of parameters formula (see [5]):*

$$x(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)Bx(s)ds, \quad \forall t \geq 0. \tag{13}$$

For $\lambda \in \rho(A)$ ($\rho(A)$ is the resolvent set of A), we consider the system (1) with $B_\lambda := \lambda R(\lambda, A_{-1})B$ instead of B. Observing that the operator B_λ is bounded, we deduce that the corresponding system admits a unique mild solution denoted by x_λ, which satisfies the following formula

$$x_\lambda(t) = S(t)x_0 - \rho \int_0^t S_{-1}(t-s)B_\lambda x_\lambda(s)ds, \quad \forall t \geq 0. \tag{14}$$

We claim that $x_\lambda(t)$ converges to $x(t)$ as $\lambda \to +\infty$. Indeed, for all $t > 0$, we have

$$\begin{aligned} x_\lambda(t) - x(t) &= \rho \int_0^t S_{-1}(t-s)B_\lambda x_\lambda(s) - \rho \int_0^t S_{-1}(t-s)Bx(s)ds \\ &= \rho \int_0^t S_{-1}(t-s)B_\lambda(x_\lambda(s) - x(s))ds + \rho \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \rho \int_0^t S_{-1}(t-s)Bx(s)ds. \end{aligned}$$

Then, using (5), this yields for all $t \in [0, T]$

$$\|x_\lambda(t) - x(t)\|_X \leq \rho M \|x_\lambda(.) - x(.)\|_{L^p(0,t;X)} + \rho \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X,$$

which by integrating gives for all $t \in [0, T]$

$$\|x_\lambda(t) - x(t)\|_{L^p(0,T;X)}^p \leq T(2\rho M)^p \|x_\lambda(.) - x(.)\|_{L^p(0,T;X)}^p +$$

$$(2\rho)^p \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X^p.$$

It follows that

$$\|x_\lambda(.) - x(.)\|_{L^p(0,T;X)}^p \leq \rho C_\rho \int_0^T \left\| \int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds \right\|_X^p dt,$$

with $C_\rho := \frac{(2\rho)^p}{1-T(2\rho M)^p}$.

It is clear that $\lim_{\lambda\to\infty}\int_0^t S_{-1}(t-s)B_\lambda x(s)ds = \int_0^t S_{-1}(t-s)Bx(s)ds$ *in X and we have*

$$\left\|\int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X \leq \left\|\int_0^t S_{-1}(t-s)B_\lambda x(s)ds\right\|_X + \left\|\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X.$$

Moreover, by the admissibility assumption we get

$$\left\|\int_0^t S_{-1}(t-s)B_\lambda x(s)ds - \int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X \leq 2M\|x(.)\|_{L^p(0,t;X)}.$$

Then, according to the dominated convergence theorem we have

$$\begin{aligned}\lim_{\lambda\to\infty}\|x_\lambda(.)-x(.)\|^p_{L^p(0,T;X)} &\leq \lim_{\lambda\to\infty}\int_0^T\left\|\rho\int_0^t S_{-1}(t-s)B_\lambda x(s)ds-\rho\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X dt\\ &= \int_0^T\lim_{\lambda\to\infty}\left\|\rho\int_0^t S_{-1}(t-s)B_\lambda x(s)ds-\rho\int_0^t S_{-1}(t-s)Bx(s)ds\right\|_X dt\\ &= 0.\end{aligned}$$

Let $x_0 \in D(A_{\rho B})$ *be fixed. Thus for all* $t>0$ *we have*

$$\frac{d}{dt}\|x_\lambda(t)\|^2_X \leq -2\rho\mathcal{R}e\langle B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X, \quad \forall t>0\cdot \tag{15}$$

For all $t>0$, *we have the following equality*

$$\begin{aligned}\langle B_\lambda S(t)x_0, S(t)x_0\rangle_X &= \langle B_\lambda S(t)x_0, S(t)x_0 - x_\lambda(t)\rangle_X\\ &+ \langle B_\lambda S(t)x_0 - B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X + \langle B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X\end{aligned}$$

which gives

$$\begin{aligned}\langle S(t)x_0, B^*\lambda R^*(\lambda,A)S(t)x_0\rangle_X &= \langle S(t)x_0, B^*\lambda R^*(\lambda,A)\,(S(t)x_0 - x_\lambda(t))\rangle_X\\ &+ \langle S(t)x_0 - x_\lambda(t), B^*\lambda R^*(\lambda,A)x_\lambda(t)\rangle_X + \langle B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X\end{aligned}$$

Let us estimate each term of this last expression.

We deduce from (i) *that for some constant* $M>0$ *and for all* $u\in L^p(0,T;X)$, *we have*

$$\left\|\int_0^T S_{-1}(T-s)Bu(s)ds\right\|_X \leq M\|u\|_{L^p(0,T;X)} \tag{16}$$

The formula (14) combined with the estimate (6), gives

$$\|S(t)x_0 - x_\lambda(t)\|_X = \rho\left\|\int_0^t S_{-1}(t-s)Bx_\lambda(s)ds\right\|_X \leq \rho M\|x_\lambda(.)\|_{L^p(0,T;X)}, \forall t\in[0,T].$$

Then, according to Lemma II.2, we conclude that

$$\|S(t)x_0 - x_\lambda(t)\|_X \leq \frac{\rho M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X \tag{17}$$

For every $t > 0$ we have, for some positive constant C,

$$\begin{array}{rcl}\mathcal{R}e\,\langle S(t)x_0, B^*\lambda R^*(\lambda, A)S(t)x_0\rangle_X & \leq & C\|S(t)x_0\|_X\|B^*\|_{\mathcal{L}(D(A^*),X)}\|\lambda R^*(\lambda, A)\|_{\mathcal{L}(X)}\|S(t)x_0 - x_\lambda(t)\|_X \\ & + & C\|B^*\|_{\mathcal{L}(D(A^*),X)}\|\lambda R^*(\lambda, A)\|_{\mathcal{L}(X)}\|x(t)\|_X\|S(t)x_0 - x(t)\|_X \\ & + & \mathcal{R}e\,\langle B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X\,,\end{array}$$

where we have taken into consideration the identification $X_{-1} \cong D(A^*)'$.
Using the fact that $S(t)$ is a contraction, it comes

$$\begin{array}{rcl}\mathcal{R}e\,\langle S(t)x_0, B^*\lambda R^*(\lambda, A)S(t)x_0\rangle_X & \leq & C\|B^*\|_{\mathcal{L}(D(A^*),X)}\|x_0\|_X\|S(t)x_0 - x_\lambda(t)\|_X \\ & + & C\|B^*\|_{\mathcal{L}(D(A^*),X)}\|x(t)\|_X\|S(t)x_0 - x(t)\|_X + \mathcal{R}e\,\langle B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X\,.\end{array}$$

Using (10) and (17), we deduce that for all $t \in (0, T]$ we have

$$\begin{array}{rcl}\mathcal{R}e\,\langle S(t)x_0, B^*\lambda R^*(\lambda, A)S(t)x_0\rangle_X & \leq & \dfrac{\rho CLMT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X^2 \\ & + & \dfrac{\rho CLMT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\left(1+\dfrac{\rho MT^{\frac{1}{p}}}{1-T^{\frac{1}{p}}\rho M}\right)\|x_0\|_X^2 + \mathcal{R}e\,\langle B_\lambda x_\lambda(t), x_\lambda(t)\rangle_X\,,\end{array}$$

with $L := \|B^*\|_{\mathcal{L}(D(A^*),X)}$. Then, integrating the last inequality and using (15) we get for all $t \in [0, t]$

$$\begin{array}{rcl}2\rho\displaystyle\int_0^T \mathcal{R}e\,\langle S(t)x_0, B^*\lambda R^*(\lambda, A)S(t)x_0\rangle_X & \leq & \dfrac{2\rho^2 CLMT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|x_0\|_X^2 \\ & + & \dfrac{2\rho^2 CLMT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\left(1+\dfrac{\rho MT^{\frac{1}{p}}}{1-T^{\frac{1}{p}}\rho M}\right)\|x_0\|_X^2 + \|x_0\|_X^2 - \|x_\lambda(T)\|_X^2,\end{array}$$

Thus, letting $\lambda \to +\infty$, we drive

$$2\mathcal{R}e\int_0^T Re\,\langle S(t)x_0, B^*S(t)x_0\rangle_X\,dt \quad \leq \quad 2\rho^2 C_1\|x_0\|_X^2 + \|x_0\|_X^2 - \|x(T)\|_X^2$$

with $C_1 = \frac{MCLT^{1+\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\left(2+\frac{\rho MT^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\right)$.
Applying the inequality (12), it follows that

$$2\rho\delta\|S(T)x_0\|_X^2 - 2\rho^2 C_1\|x_0\|_X^2 \leq \|x_0\|_X^2 - \|x(T)\|_X^2 \tag{18}$$

Using Lemma II.2, we deduce via the variation of constants formula (13) that for all $t \in [T, 2T]$, we have

$$\begin{array}{rl}\|x(t)\|_X & \leq \|S(T)x_0\|_X + \rho\|\int_0^t S_{-1}(t-s)Bx(s)ds\|_X \\ & \leq \|S(T)x_0\|_X + \rho M_\rho\|x_0\|_X.\end{array}$$

By reiterating the processes for $t \in [kT, (k+1)T], k \geq 1$, we deduce that

$$\|x(t)\|_X \leq \|S(T)x(kT)\|_X + \rho M_\rho\|x(kT)\|_X.$$

Then for all $k \geq 1$, we have

$$\|x((k+1)T)\|_X^2 \leq 2\|S(T)x(kT)\|_X^2 + 2\rho M_\rho \|x(kT)\|_X^2. \tag{19}$$

Moreover, (18) becomes

$$2\rho\delta \|S(T)(kT)\|_X^2 - 2\rho^2 C_1 \|x(kT)\|_X^2 \leq \|x(kT)\|_X^2 - \|x((k+1)T)\|_X^2 \tag{20}$$

This together with (19) implies

$$\rho\delta\bigg(\|x((k+1)T)\|_X^2 - 2\rho M_\rho \|x(kT)\|_X^2\bigg) - 2C_1\rho^2 \|x(kT)\|_X^2 \leq$$

$$\|x(kT)\|_X^2 - \|x((k+1)T)\|_X^2.$$

Hence

$$(1+\rho\delta)\|x((k+1)T)\|_X^2 \leq \bigg(2\delta\rho^2 M_\rho + 2C_1\rho^2 + 1\bigg)\|x(kT)\|_X^2,\ k \geq 0\cdot$$

This implies

$$\|x\left((k+1)T\right)\|_X^2 \leq C_2 \|x(kT)\|_X^2$$

where $C_2 = \frac{2\rho^2(\delta M_\rho + C_1)+1}{1+\rho\delta}$, which is in $(0,1)$ for $\rho \to 0^+$.
Since $\|x(t)\|_X$ decreases, we get for $k = E\left(\frac{t}{T}\right)$ (where $E(.)$ is the integer part function).

$$\|x(t)\|_X^2 \leq (C_2)^k \|x_0\|_X^2,$$

which gives the following exponential decay

$$\|x(t)\|_X \leq K e^{-\sigma t}\|x_0\|,\ \forall t \geq 0\cdot$$

where $K = (C_2)^{-\frac{1}{2}}$ and $\sigma = \frac{-ln(C_2)}{2T}$. This estimate extends by density to all $x_0 \in X$. Hence the uniform exponential stability hold for any $0 < \rho < \rho_1$, where ρ_1 is such that $0 < \rho_1 < \frac{1}{T^{\frac{1}{p}} M}$ and $\frac{2\rho^2(\delta M_\rho + C_1)+1}{1+\rho\delta} \in (0,1)$.

Remark 1
In the case where $Range(BS(t)) \subset X,\ \forall t > 0$, the condition (12) is equivalent to the conventional one (see [2, 15]):

$$\int_0^T Re\,\langle BS(t)x, S(t)x\rangle_X\, dt \geq \delta \|S(T)x\|_X^2,\ \ \forall x \in X. \tag{21}$$

Moreover, the condition (21) can be weakened if an appropriate decomposition of $Range(B)$ is available. This is the aim of the next section.

iii. A range decomposition method

Let $\mathfrak{X} \oplus \mathfrak{X}_{-1}$ be a direct sum in X_{-1}, where $\mathfrak{X} = i(X)$ (i being the canonical injection of X in X_{-1}), so we can write $\mathfrak{X} = X$. Then for any $C \in \mathcal{L}(X, X_{-1})$ such that $rg(C) \subset \mathfrak{X} \oplus \mathfrak{X}_{-1}$, we set ${}_X C =: P_{\mathfrak{X}} C$, where $P_{\mathfrak{X}}$ is the projection of $\mathfrak{X}$ according to $\mathfrak{X} \oplus \mathfrak{X}_{-1}$. Now, given a pair of operators $(K, L) \in \mathcal{L}(X, X_{-1}) \times \mathcal{L}(X, X_{-1})$, the decomposition $\mathfrak{X} \oplus \mathfrak{X}_{-1}$ is said to be admissible for (K, L) if the three following properties hold:
(a) $rg(K) \subset \mathfrak{X} \oplus \mathfrak{X}_{-1}$ and $rg(L) \subset \mathfrak{X} \oplus \mathfrak{X}_{-1}$,

(b) ${}_X K$ is dissipative on $D((K+L)|_X) := \{x \in X : Kx + Lx \in X\}$,
(c) ${}_X L \in \mathcal{L}(X)$.
For our stabilization problem, we will be interested with admissible decompositions for the pairs $(A_{-1}, -\rho B)$ with $\rho > 0$ small enough. Note that if the domain of the operator $(A_{\rho B})|_X$ is independent of $\rho > 0$ (small enough), which is equivalent to $D((A_{\rho B})|_X) = D(A) \cap D(B|_X)$, then for the sum $\mathfrak{X} \oplus \mathfrak{X}_{-1}$ to be admissible for the pairs $(A_{-1}, -\rho B)$, $\rho > 0$, it suffices to be admissible for the pair (A_{-1}, B).

We are ready to state our second main result.

Theorem II.4 *Let A be the infinitesimal generator of a linear C_0-semigroup of contractions $S(t)$ on X and let $B \in \mathcal{L}(X, X_{-1})$. Let $\mathfrak{X} \oplus \mathfrak{X}_{-1}$ be an admissible decomposition for the pair $(A_{-1}, -\rho B)$ for any $\rho > 0$ small enough, and assume that for some $T > 0$, the operator B is $p-$admissible for some $1 < p < \infty$ and satisfies the estimate:*

$$\int_0^T Re\, \langle {}_X BS(t)x, S(t)x\rangle_X \, dt \geq \delta \|S(T)x\|_X^2, \ \ \forall x \in X, \tag{22}$$

for some $T, \delta > 0$.
Then there is a $\rho_1 > 0$ such that the system (1) is exponentially stable on X for all $\rho \in (0, \rho_1)$.

Proof 3 *Let $0 < \rho < \frac{1}{T^{\frac{1}{p}} M}$, and let $x(t)$ be the unique mild solution of the system (1) given for $x_0 \in D((A_{\rho B})|_X)$ by the formula (13).*
The admissibility assumption on B together with Lemma II.2 implies the following estimate for $t \in [0, T]$:

$$\|x(t) - S(t)x_0\|_X \leq \frac{\rho M T^{\frac{1}{p}}}{1 - \rho T^{\frac{1}{p}} M} \|x_0\|_X \tag{23}$$

Moreover, observing that $A_{\rho B}x(t) = {}_X(A_{\rho B})x(t)$, we can write

$$\frac{d}{dt}\|x(t)\|_X^2 = 2\mathcal{R}e\langle {}_X(A_{-1})x(t) - \rho\ {}_X Bx(t), x(t)\rangle_X, \ \forall t > 0\cdot$$

Integrating this last equality and using the dissipativeness of ${}_X(A_{-1})$ gives

$$2\rho \int_s^t \mathcal{R}e\langle {}_X Bx(\tau), x(\tau)\rangle_X d\tau \leq \|x(s)\|_X^2 - \|x(t)\|_X^2, \ t \geq s \geq 0. \tag{24}$$

We have the following equality

$$\begin{array}{rcl} \langle {}_X BS(t)x_0, S(t)x_0\rangle_X & = & \langle {}_X BS(t)x_0 - \ {}_X Bx(t), S(t)x_0\rangle_X \\ & + & \langle {}_X Bx(t), S(t)x_0 - x(t)\rangle_X + \langle {}_X Bx(t), x(t)\rangle_X \cdot \end{array}$$

Then using the fact that the operator ${}_X B$ is bounded, it comes

$$\begin{array}{rcl} \mathcal{R}e\, \langle {}_X BS(t)x_0, S(t)x_0\rangle_X & \leq & \|{}_X B\|_{\mathcal{L}(X)} \|x_0\|_X \|S(t)x_0 - x(t)\|_X \\ & + & \|{}_X B\|_{\mathcal{L}(X)} \|x(t)\|_X \|S(t)x_0 - x(t)\|_X + \mathcal{R}e\, \langle {}_X Bx(t), x(t)\rangle_X \end{array}$$

The estimate (23) combined with (10), implies

$$\begin{aligned}\mathcal{R}e\,\langle_X BS(t)x_0, S(t)x_0\rangle_X &\leq \|_X B\|_{\mathcal{L}(X)} \frac{\rho M T^{\frac{1}{p}}}{1-T^{\frac{1}{p}}\rho M}\|x_0\|_X^2 \\ &+ \|_X B\|_{\mathcal{L}(X)} \frac{\rho M T^{\frac{1}{p}}}{1-T^{\frac{1}{p}}\rho M}\left(1+\frac{\rho M T^{\frac{1}{p}}}{1-T^{\frac{1}{p}}\rho M}\right)\|x_0\|_X^2 \\ &+ \mathcal{R}e\,\langle_X Bx(t), x(t)\rangle_X\,,\ \forall t\in[0,T]\cdot\end{aligned}$$

Integrating this inequality and using the inequality (22), we deduce that

$$\delta\|S(T)x(kT)\|_X^2 - \rho C_1\|x(kT)\|_X^2 \leq \int_{kT}^{(k+1)T} \mathcal{R}e\,\langle_X Bx(s), x(s)\rangle_X\, ds$$

with $C_1 = \frac{MT^{1+\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\|_X B\|_{\mathcal{L}(X)}\left(2+\frac{\rho M T^{\frac{1}{p}}}{1-\rho T^{\frac{1}{p}}M}\right)$.
By using Lemma II.2, we derive

$$\rho\delta\left(\|x((k+1)T)\|_X^2 - 2\rho M_\rho\|x(kT)\|_X^2\right) - 2C_1\rho^2\|x(kT)\|_X^2 \leq$$

$$\|x(kT)\|_X^2 - \|x((k+1)T)\|_X^2,$$

or equivalently

$$\|x\left((k+1)T\right)\|_X^2 \leq C_2\|x(kT)\|_X^2,$$

where $C_2 = \frac{2\rho^2(\delta M_\rho + C_1)+1}{1+\rho\delta}$, *which lies in* $(0,1)$ *for* $\rho \to 0^+$.
Hence, using the decreasing of $\|x(t)\|_X$, *we deduce the following exponential decay*

$$\|x(t)\|_X \leq Ke^{-\sigma t}\|x_0\|,\ \forall t\geq 0,$$

where $K = (C_2)^{-\frac{1}{2}}$ *and* $\sigma = \frac{-ln(C_2)}{2T}$. *This estimate extends by density to all* $x_0 \in X$. *Thus, taking* $\rho_1 > 0$ *such that* $0 < \rho_1 < \frac{1}{T^{\frac{1}{p}}M}$ *and* $C_2 \in (0,1)$, *we get the result of the theorem.*

III. Examples

Example 1 *Let* Ω *be an open and bounded subset of* $\mathbf{R}^d$, $d \geq 1$, *and let us consider the following bilinear equation of diffusion type*

$$\begin{cases}\frac{\partial}{\partial t}x = \Delta x + gx + v(t)(-\Delta)^{\frac{1}{2}}x & in\ \ \Omega\times(0,\infty),\\ x(t) = 0 & on\ \ \partial\Omega\times(0,\infty),\\ x(0) = x_0 & in\ \ \Omega.\end{cases} \tag{25}$$

where $g \in L^\infty(\Omega)$, v *is a real valued bilinear control and* $x(t) = x(\zeta,t) \in L^2(\Omega)$ *is the state. The system (25) is an example of fractional equation of diffusion equations type, and may describe transport processes in complex systems which are slower than the Brownian diffusion. As practical situations displaying such anomalous behaviour, let us mention the charge carrier transport in amorphous semiconductors, the nuclear magnetic resonance diffusometry in percolative and porous media etc (see [3, 10, 13, 12]). Here, we aim* *to show the exponential stabilization of (25). Let us observe that system (25) can be written in the form of (1) if we close it by the switching feedback control* $v(t) = -\rho \mathbf{1}_{\{t\geq 0\ /\ x(t)\neq 0\}}$. *This is because we have* $\mathbf{1}_{\{t\geq 0\ /\ x(t)\neq 0\}}(-\Delta)^{\frac{1}{2}}x(t) = (-\Delta)^{\frac{1}{2}}x(t),\ \forall t \geq 0.$. *Let us take the state space* $X = L^2(\Omega)$ *(endowed with its natural scalar product* $\langle\cdot,\cdot\rangle_X$*), and consider the control operator* $B = (-\Delta)^{\frac{1}{2}}$ *and the system's operator* $A = \Delta + gI$ *with* $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. *The operator* A *generates an analytic semigroup* $S(t)$ *on* X *(see [6], p. 107 and p. 176) which is given by the following variation of constants formula:*

$$S(t)x = S_0(t)x + \int_0^t S_0(t-s)g(\xi)S(s)xds, \ \ t \geq 0,$$

where $S_0(t)$ *is the semigroup generated by* A *with* $g = 0$.
Let us verify the assumptions of Theorem 2. *In order to make the computation easier, we restrict our self to the mono-dimension case, thus we consider* $\Omega = (0,1)$. *In this case the semigroup* $(S_0(t))$ *is is given by*

$$S_0(t)x = \sum_{j\geq 1} e^{-\alpha_j t}\langle x, \phi_j\rangle_X\ \phi_j,\ \forall x \in L^2(\Omega)$$

with $\alpha_j = j^2\pi^2, j \geq 1$ *is the set of eigenvalues of* $-\Delta$ *with the corresponding orthonormal basis of* $L^2(\Omega)$: $\phi_j(x) = \sqrt{2}\sin(j\pi x)$. *Moreover, the semigroup* $S(t)$ *is a contraction if in addition*

$$\int_\Omega g(\xi)y^2(\xi)d\xi \leq \|y\|^2_{H_0^1(\Omega)},\ \forall y \in H_0^1(\Omega).$$

Thus, in the sequel we suppose this condition satisfied. Then the operator B *can be expressed as*

$$Bx = \sum_{j\geq 1} \alpha_j^{\frac{1}{2}}\langle x, \phi_j\rangle_X\ \phi_j,\ x \in L^2(\Omega).$$

Here, B *is unbounded on* $L^2(\Omega)$ *and it is bounded from* $L^2(\Omega)$ *onto the space* X_{-1} *defined as the completion of* $L^2(\Omega)$ *for the norm* $\|y\| = \big(\sum_{j\geq 1}\frac{1}{\alpha_j}\langle y, \phi_j\rangle^2\big)^{\frac{1}{2}},\ \forall y \in L^2(\Omega)$, *which can be also interpreted as the dual space of* $D((-\Delta)^{\frac{1}{2}})$ *with respect to the* $L^2(\Omega)-$*topology (the space* $L^2(\Omega)$ *being the pivot space). Note also that the space* $D((-\Delta)^{\frac{1}{2}})$ *can be doted with the norm* $\|x\|_{D((-\Delta)^{\frac{1}{2}})} = \left(\sum_{j\geq 1}\alpha_j|\langle x, \phi_j\rangle_X|^2\right)^{\frac{1}{2}}$.

Let $p > 1$, $T > 0$ *and let* $u \in L^p(0, T; X)$. *It follows from the fact that* $(-\Delta)^{\frac{1}{2}} \in \mathcal{L}(X, X_{-1})$, *that the* $X_{-1}-$ *valued integral* $\int_0^T S_{-1}(T-s)(-\Delta)^{\frac{1}{2}}u(s)ds$ *is well-defined (see [6], Theorem 5.34). Moreover, since the semigroup* $(S(t))_{t\geq 0}$ *is analytic, then so is* $((S_{-1}(t))_{t\geq 0}$. *This implies that* $S_{-1}(\frac{T-s}{2})(-\Delta)^{\frac{1}{2}}u(s) \in X,\ \forall s \in [0, T)$ *(see [6], p. 101). Then we have* $\displaystyle\int_0^T S_{-1}(T-s)(-\Delta)^{\frac{1}{2}}u(s)ds \in X$, *which gives the* $p-$*admissibility of* B *(see [14], Prop. 3.3 and [17], Lemma. 4.3.9).*
For all $x \in L^2(\Omega)$, $t \geq 0$ *and* $j \geq 1$, *we have*

$$\begin{aligned} |\langle\int_0^t S_0(t-s)gS(s)xds, \phi_j\rangle_X| &= \int_0^t\langle S_0(t-s)gS(s)x, \phi_j\rangle ds = |\int_0^t\langle gS(s)x, e^{-\alpha_j(t-s)}\phi_j\rangle_X ds| \\ &\leq \|g\|_{L^\infty(\Omega)}\|x\|_X\,\frac{1-e^{-\alpha_j t}}{\alpha_j}. \end{aligned}$$

We deduce that

$$|\langle S(t)x, \phi_j\rangle_X| \leq e^{-\alpha_j t}\|x\|_X + \|g\|_{L^\infty(\Omega)}\ \frac{1-e^{-\alpha_j t}}{\alpha_j}\ \|x\|_X,\ t \geq 0,\ j \geq 1. \tag{26}$$

Now for any $x \in X$, we have

$$BS(t)x = \sum_{j\geq 1} \alpha_j^{\frac{1}{2}} \left\langle S(t)x, \phi_j \right\rangle_X \phi_j.$$

This combined with (26) implies that $BS(t)x \in X$ for all $x \in X$ and for any $t > 0$.
Now, using the series expansion of $BS(t)x$ for $x \in X$, we get

$$\begin{aligned} \langle BS(t)x, S(t)x \rangle_X &= \sum_{j\geq 1} \alpha_j^{\frac{1}{2}} \left\langle S(t)x, \phi_j \right\rangle_X^2 \\ &\geq \|S(t)x\|_X^2 \\ &\geq \|S(T)x\|_X^2, \ \forall t \in [0, T]. \end{aligned}$$

It follows that the assumption (12) is fulfilled.
We conclude by Theorem II.3 that for $\rho > 0$ small enough, the control $v(t) = -\rho \mathbf{1}_{\{t\geq 0,\ x(t)\neq 0\}}$ guarantees the uniform exponential stability of the system (25).

Example 2 *Consider the following system*

$$(S_0) \begin{cases} \frac{\partial}{\partial t}(\zeta, t) = \frac{\partial}{\partial \zeta} x(\zeta, t) - \alpha x(\zeta, t) + v(t)h(\zeta)x(\zeta, t) & \text{in } (0,1) \times (0, \infty), \\ x(1, t) = 0 & \text{in } (0, \infty), \\ x(\cdot, 0) = x_0 \in L^2(0, 1) \end{cases}$$

where $X = L^2(0,1)$, $\alpha > 0$ and $h \in L^\infty(0,1)$ is such that $h \geq c > 0$, for some positive constant c. Here we can take $A = \frac{d}{d\zeta} - \alpha\, id$ with domain $D(A) := \left\{x \in H^1(0,1) : x(1) = 0\right\}$.
The operator A is the generator of a contraction semigroup $(S(t)_{t\geq 0})$ given by

$$\big(S(t)x\big)(\zeta) = \begin{cases} e^{-\alpha t} x(\zeta + t) & \text{if } \zeta + t \leq 1, \\ 0 & \text{else.} \end{cases}$$

According to previous theorems, the system (S_0) is exponentially stablilizable by the switching feedback control $v(t) = -\rho \mathbf{1}_{\{t\geq 0\ /\ x(t)\neq 0\}}$. Indeed, here the semigroup $S(t)$ is a contraction (so that $\|S(t)\|$ is decreasing) and the linear bounded operator $B_1 := h\, id$ is a bounded linear operator ($h \in L^\infty$) and satisfies the observation condition (since $h \geq c > 0$). Let us now consider the following system

$$(S_1) \begin{cases} \dot{x}(t) = x_\zeta(t) - \alpha x(\zeta, t) + v(t)h(\zeta)x(t) & \text{in } (0,1) \times (0, \infty) \\ x(1, t) + \epsilon\psi(x(t)) = 0 & \text{in } (0, \infty) \end{cases}$$

where $\psi : X \to \mathbf{R}$ is a non null linear functional of X. This may be seen as a perturbed version of (S_0) on its boundary conditions. According to Riesz representation, one can assume that $\psi(x) = \int_0^1 f(s)x(s)ds,\ \forall x \in X$ for some $f \in X - (0)$.
We aim to show that under small valuers of $\epsilon > 0$, this system is still exponentially stabilizable.
The system (S_1) can be reformulated as:

$$(S_2) \begin{cases} \dot{x}(t) = \mathcal{A}x(t) + v(t)h(\zeta)x(t) & \text{in } (0,1) \times (0, \infty) \\ x(0) = x_0 & \text{in } (0,1) \end{cases}$$

where $\mathcal{A} : D(\mathcal{A}) \subset X \to X$ is defined by:

$$\mathcal{A}x := Ax - \epsilon h x,\ \forall x \in D(\mathcal{A}) := \Big\{x \in H^1(0,1), x(1) + \epsilon\psi(x) = 0\Big\}.$$

We claim that $\mathcal{A}$ is the generator of a strongly continuous semigroup on X. In order to verify this assertion, we will consider $\mathcal{A}$ as a perturbation of the generator A.
In order to write the system (S_2) in the form (1), let us consider the function $\theta(\zeta) = \mathbf{1}(\zeta) := 1,\ \zeta \in X$, which is such that $A_m\theta = 0$, and $\theta(1) = 1$, where $A_m := \frac{d}{d\zeta}$ with domain $D(A_m) := H^1(0,1)$.
Let us introduce the following operator

$$Bx = hx - \psi(x)A_{-1}\theta, \quad \forall x \in X$$

which is a one to one operator since we have $\theta \notin D(A)$.
In the sequel, we will verify the assumptions of Theorem II.4 and then conclude the stabilization of the perturbed system (S_1).
• *From the boundary conditions of (S_2), we can see that*

$$\forall x \in X,\ x \in D(\mathcal{A}) \Leftrightarrow x \in H^1(0,1) \text{ and } x + \epsilon\psi(x)\theta \in D(A)\cdot$$

This together with the definition of θ implies that for $x \in D(\mathcal{A})$, we have

$$\begin{array}{rcl} X \ni \mathcal{A}x &=& A_m x - \epsilon h x \\ &=& A_m\big(x + \epsilon\psi(x)\theta\big) - \epsilon h x \\ &=& A\big(x + \epsilon\psi(x)\theta\big) - \epsilon h x \\ &=& A_{-1}\big(x + \epsilon\psi(x)\theta\big) - \epsilon h x \\ &=& A_{-1}x - \epsilon B x \\ &=& \big(A_{-1} - \epsilon B\big)|_X x \end{array}$$

Moreover, for all $x \in D((A_{-1} - \epsilon B)|_X)$, we have $A_{-1}\big(x + \epsilon\psi(x)\theta\big) \in X$, i.e. $x + \epsilon\psi(x)\theta \in D(A) \subset H^1(0,1)$ which implies that $x \in H^1(0,1)$. Then we have $(A_{-1} - \epsilon B\big)|_X x = \mathcal{A}x$. In other words,

$$\big(\mathcal{A}, D(\mathcal{A})\big) = \big((A_{-1} - \epsilon B)|_X, D(A_{-1} - \epsilon B)|_X)\big).$$

• *The operator $(A_{-1} - \epsilon B)_X$ is a generator if we can show that*

$$\int_0^1 S_{-1}(1-r)\psi(u(r))A_{-1}\theta dr \in X,$$

or, equivalently

$$\int_0^1 S_{-1}(1-r)\mathbf{1}(.)\psi(u(r))dr \in D(A),\ \forall u \in L^2(0,1;X).$$

We have

$$\begin{array}{rcl} \displaystyle\int_0^1 S_{-1}(1-r)\mathbf{1}(.)\psi(u(r))dr &=& \displaystyle\int_0^1 \psi(u(r))S(1-r)\mathbf{1}(\cdot)dr \\ &=& \displaystyle\int_.^1 e^{-\alpha(1-r)}\psi(u(r))dr := g(.). \end{array}$$

Since $\psi o u \in L^2(0,1)$, this implies that $g \in H^1(0,1)$ and $g(1) = 0$. In other words, $g \in D(A)$. Hence, for $\epsilon > 0$ small enough, the system (S_1) is well-posed.
• *Here we can take $\mathfrak{X}_{-1} = span(A_{-1}\theta)$, so we obtain an admissible decomposition for the pair $(A_{-1}, -\epsilon B)$. Indeed, it is clear that ${}_X B x = hx,\ x \in X$, so ${}_X B$ is a bounded operator from X to X.*
Moreover, for all $x \in D((A_{-1} - \epsilon B)|_X) = D\big((A_{-1} + \epsilon\psi(.)A_{-1}\theta)|_X\big)$, we have

$$A_{-1}x = A_{-1}(x + \epsilon\psi(x)\theta) - \epsilon\psi(x)A_{-1}\theta = A(x + \epsilon\psi(x)\theta) - \epsilon\psi(x)A_{-1}\theta,$$

from which it comes that

$$ {}_{X}(A_{-1})x = A(x + \epsilon\psi(x)\theta), \ \forall x \in D((A_{-1} - \epsilon B)|_X), $$

where

$$ D((A_{-1} - \epsilon B)|_X) = \{x \in L^2(0,1) \ / \ x + \epsilon\psi(x)\theta \in D(A)\} $$

Then for $x \in D\big((A_{-1} - \epsilon B)|_X\big)$, *we have* $(A_{-1} - \epsilon B)x \in X$ *or equivalently* $x + \epsilon\psi(x)\theta \in D(A)$, *and*

$$ \begin{aligned} \langle {}_{X}(A_{-1})x, x\rangle &= \langle A(x + \epsilon\psi(x)\theta), x\rangle \\ &= \langle A_m(x + \epsilon\psi(x)\theta), x\rangle \\ &= \langle A_m x, x\rangle \\ &= \int_0^1 x'(s)x(s)ds - \alpha\|x\|^2 \\ &\leq \big(\frac{\epsilon^2\|f\|^2}{2} - \alpha\big)\|x\|^2 - \frac{1}{2}x^2(0)\cdot \end{aligned} $$

Thus the operator ${}_{X}(A_{-1})$ *is dissipative in* $D((A_{-1} - \epsilon B)|_X)$ *for every* $0 < \epsilon \leq \frac{\left(2\alpha\right)^{1/2}}{\|f\|}$.
• *Finally, the observation estimate follows from the fact that* $h \geq c > 0$ *and that for any* $x \in X$, *the mapping* $t \mapsto \|S(t)x\|$ *is decreasing.*
We conclude by Theorem II.4 that for $\epsilon > 0$ *small enough, the control* $v(t) = -\epsilon\mathbf{1}_{\{t\geq 0:\, x(t)\neq 0\}}$ *guarantees the exponentially stabilization of the system* (S_1).

IV. Conclusion

In this paper we have shown that it is possible for a linear system with dissipative dynamic, to be exponentially stable under small Desch-Schapacher perturbations of the dynamic. The main assumptions of sufficiency are formulated in terms of admissibility and observability assumptions of unbounded linear operators. An explicit decay rate of the stabilized state is given. The previous research on this problem concerned either bounded or Miyadera's type perturbations [11, 15]. The main stabilization result is further applied to show the uniform exponential stabilization of unbounded bilinear reaction diffusion and transport equations using a bang bang controller.

Conflict of interest

The authors declare that they have no conflict of interest.

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