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Dependence on parameters for nonlinear equations – abstract principles and applications

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Summary

We provide parameter dependent version of the Browder–Minty Theorem in case when the solution is unique utilizing different types of monotonicity and compactness assumptions related to condition $(S)_2$. Potential equations and the convergence of their Euler action functionals is also investigated. Applications towards the dependence on parameters for both potential and non-potenial nonlinear Dirichlet boundary problems are given.

KEYWORDS:
Browder-Minty Theorem, dependence on parameters, Dirichlet problem, monotone operator

1 | INTRODUCTION

Let (X, d) be a complete metric space and (Y, ϱ) be a metric space. The following well known result provides a type of continuous dependence on parameters for fixed points obtained in the Banach Contraction Principle, see [1].

Theorem 1 (*Parameter Contraction Principle*)**.** Assume $T: X \times Y \to X$ is a continuous function and that there exists a continuous function $L: Y \to [0, 1)$ such that

$$d\big(T(x_1, y), T(x_2, y)\big) \le L(y)d(x_1, x_2) \quad \text{for all } x_1, x_2 \in X,\ y \in Y. \tag{1}$$

Then, for every fixed $y \in Y$, the map $x \mapsto T(x, y)$ has a unique fixed point $\Phi(y)$. Moreover, the function $y \mapsto \Phi(y)$ is continuous.

Note that in [1] it is assumed that $0 \le L < 1$ is a constant. This is why we give a proof of Theorem 1 in the Appendix. Note that the uniqueness of the fixed point is crucial here and will be crucial in what follows in this work.

The assumptions of Theorem 1 may be relaxed as follows: Let $T: X \times Y \to X$ be continuous on Y and a contraction on X uniformly in Y. Then T is in fact continuous on $X \times Y$ into X. If T is Lipschitz on Y uniformly in X then Φ is Lipschitz continuous itself. Note that this observation is only a minor technical improvement since the core of this result lies in the fact that T is Lipschitz on Y uniformly in X.

Theorem 1 is of importance for applications for example towards the Dirichlet problems and suggests that other parameter dependent existence principles may be derived basing on other existence and uniqueness principles. Our paper is therefore concerned with derivation of general existence and depending on parameters principles related to the theory of monotone operators as well as the direct variational method. Applications to boundary value problems and next for various types of differential operators are given. The simple application of Theorem 1 suggesting what we are concerned with in this work is as follows. Let $X = C[0, 1]$. Let $\left(w_n\right)_{n=1}^{\infty} \subset X$ be a sequence of parameters, $w_n \rightrightarrows w_0$, and let us consider the following family of Dirichlet problems for $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$:

$$\begin{cases} -\ddot{u}(t) = f\left(t, u(t), w_n(t)\right), \text{ for } t \in (0, 1), \\ u(0) = u(1) = 0 \end{cases} \tag{2}$$

under condition

$$\left.\begin{array}{l} f:\ [0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R} \text{ is a continuous function which is Lipschitz with respect to a second} \\ \text{variable, that is there is a constant } L\in[0,1) \text{ such that} \\ \qquad\qquad |f(t,x,w)-f(t,y,w)|\le L\,|x-y| \\ \text{for all } t\in[0,1] \text{ and } w,x,y\in\mathbb{R}. \end{array}\right\} \tag{L}$$

The result the dependence on parameters for problem (2) now follows:

Theorem 2. Assume that condition (L) holds. For each $n\in\mathbb{N}_0$ there exists exactly one solution $u_0\in C^2[0,1]$ to (2) corresponding to w_n. Moreover, $u_n\rightrightarrows u_0$ on $[0,1]$.

Proof. We apply Theorem 1. Let define $T:\ X\times X\to X$ by

$$T(u)(t)=\int_0^1 G(t,s)\,f(s,u(s),w(s))\,ds$$

for $(u,w)\in X\times X$, where $G:\ [0,1]\times[0,1]\to\mathbb{R}$ is the Green function for the Dirichlet boundary value problem, namely

$$G(t,s)=\begin{cases} s(1-t) & \text{if } 0\le s\le t\le 1,\\ t(1-s) & \text{if } 0\le t\le s\le 1.\end{cases}$$

By assumption (L) we see that T satisfies (1). Thus it remains to show that T is continuous. Take sequence $\left(w_n\right)_{n=1}^{\infty}, \left(u_n\right)_{n=1}^{\infty}\subset X$ such that $w_n\rightrightarrows w_0$ and $u_n\rightrightarrows u_0$ on $[0,1]$. Since f is continuous it follows that

$$f\left(\cdot,u_n(\cdot),w_n(\cdot)\right)\rightrightarrows f\left(\cdot,u_0(\cdot),w_0(\cdot)\right)$$

on $[0,1]$. The continuity of T is now immediate. □

When the parameter enters the equation in a linear manner we see that no other assumptions than does leading to the application of the Banach Contraction Principle are required.

Remark 1. If instead of (2) we consider

$$\begin{cases} -\ddot{u}(t)=f(u(t))+g(t)\,w_n(t), & \text{for } t\in(0,1),\\ u(0)=u(1)=0, \end{cases} \tag{3}$$

where $f:\ \mathbb{R}\to\mathbb{R}$ is a contraction and $g\in C[0,1]$, then for each $n\in\mathbb{N}_0$ there exists exactly one solution $u_0\in C^2[0,1]\cap C_0[0,1]$ to (3) corresponding to w_n. As before $u_n\rightrightarrows u_0$ on $[0,1]$.

The above result follows easily from a direct form of operator T which is due to the fact that we work with ordinary differential equation. For the PDE case we would have to apply some different procedure by using an inverse of a differential operator, as is described in [1]. The above result which is in fact about the uniqueness, the existence of a solution and the dependence of a solution on (functional) parameters best reflects what is known as the Hadamard programme to which we wish to contribute here as well. There are many results concerning the existence and multiplicity, a fewer about uniqueness and much less dealing with the dependence on parameters. As illustrated by Theorem 1 dependence on parameters follows from main existence principles given some uniform type of underlying assumptions leading to the boundedness of the sequence of solutions corresponding to the sequence of parameters. Despite this fact, the approach towards the continuous dependence on parameters has not been focused on developing some abstract principles. Instead, each problem have been treated separately. Paper [2] describes the approach by the theory of monotone operators for elliptic problems on the Sierpinski Gasket, the variational approach towards the dependence on parameters together with some application to optimal control problems governed by the Dirichlet problem is given in [3], where the direct variational method is again utilized. Some abstract scheme working for semilinear problems and based on a dual least action principle is to be found in [4] as an attempt to provide some general scheme. We also mention the following papers which use the idea of a dependence on parameters as a separate result, as in [5], or else as a toll applied in the proofs of main results, as in [6,7].

Being inspired by Theorem 1 we decided to investigate whether some other celebrated existence principles involving also uniqueness like the Minty–Browder Theorem and the Direct Variational Method (namely the Weierstrass Theorem) have their parameter dependent variants which would work for diverse boundary value problems. In the proofs the elements of the theory of monotone operators are used (we follow[8,9], for some background) as well as some optimization tricks (see for example[10]). Paper is organized as follows: In Section 2 we recall some necessary background providing detailed comments and diagrams when necessary; in Section 3 we concentrate on derivation of a parametric version of the Browder–Minty Theorem describing various types of assumptions which could be applied and relating them one to another; Section 4 is devoted to some applications. In the Appendix we include some proofs of parametric counterparts of background results used in the paper. The proofs easily follow from those given in the literature, but since these are not commonly included in background sources we decided to provide them here.

2 | RESULT BY THE THEORY OF MONOTONE OPERATORS

2.1 | Preliminary setting

If not said otherwise, E is a real, separable and reflexive Banach space. The norm in E is denoted by $\|\cdot\|$ and in E^* by $\|\cdot\|_*$. The assumption about separability can be avoided with some technical difficulties in the main theorem on monotone operators, see[11] but since in our applications the spaces are separable, we decided not to complicate our approach with to much technical details. We recall what is necessary about the theory of monotone operators in what follows.

Operator $A: E \to E^*$ is called:

- *monotone* if for any $u, v \in E$ it holds
$$\langle A(u) - A(v), u - v\rangle \geq 0;$$
- *strictly monotone* if for any distinct $u, v \in E$ it holds
$$\langle A(u) - A(v), u - v\rangle > 0;$$
- m-strongly monotone or strongly monotone (with a constant m), if there exists $m > 0$ such that any $u, v \in E$ it holds
$$\langle A(u) - A(v), u - v\rangle \geq m\|u - v\|^2$$
- *uniformly monotone* if there exists an increasing function $\rho: [0, +\infty) \to [0, +\infty)$ such that $\rho(0) = 0$ and for all $u, v \in E$
$$\langle A(u) - A(v), u - v\rangle \geq \|u - v\|\,\rho(\|u - v\|);$$
- *d-monotone* if for some increasing function $\rho: [0, +\infty) \to \mathbb{R}$ it holds for all $u, v \in E$
$$\langle A(u) - A(v), u - v\rangle \geq \big(\rho(\|u\|) - \rho(\|v\|)\big)(\|u\| - \|v\|). \tag{4}$$

When E is strictly convex, it follows that a d-monotone operator is strictly monotone. A strongly monotone operator is obviously uniformly monotone and uniformly monotone is also strictly monotone.

We recall some basic properties, which provide regularity of an inverse operator. We say that operator $A: E \to E^*$ satisfies:

- *condition $(S)_+$*, if
$$\left.\begin{array}{r} u_n \rightharpoonup u_0 \\ \limsup\limits_{n\to\infty}\left\langle A(u_n) - A(u_0), u_n - u_0\right\rangle \leq 0 \end{array}\right\} \Rightarrow u_n \to u_0;$$
- *condition (S)*, if
$$\left.\begin{array}{r} u_n \rightharpoonup u_0 \\ \left\langle A(u_n) - A(u_0), u_n - u_0\right\rangle \to 0 \end{array}\right\} \Rightarrow u_n \to u_0;$$
- *condition $(S)_0$*, if
$$\left.\begin{array}{r} u_n \rightharpoonup u_0 \\ A(u_n) \rightharpoonup f \\ \langle A(u_n), u_n\rangle \to \langle f, u_0\rangle \end{array}\right\} \Rightarrow u_n \to u_0;$$
- *condition $(S)_2$*, if
$$\left.\begin{array}{r} u_n \rightharpoonup u_0 \\ A(u_n) \to A(u_0) \end{array}\right\} \Rightarrow u_n \to u_0;$$

While a uniformly monotone operator satisfies condition $(S)_+$, a d-monotone operator does so in case E is additionally uniformly convex or more generally in a space which has *the Kadec–Klee property*, that is if weak and strong convergence coincides on the unit sphere $\{u \in E : \|u\| = 1\}$. Strongly continuous (weak - to - strong) perturbations of operators do not violate the $(S)_+$. Note that a strongly continuous operator is necessarily compact, that is continuous and sends bounded sets into compact ones.

We say that $A : E \to E^*$ is called *coercive* when there is a function $\gamma : [0, +\infty) \to \mathbb{R}$ with $\lim_{x\to+\infty} \gamma(x) = +\infty$ such that

$$\langle A(u), u\rangle \geq \gamma(\|u\|)\,\|u\|.$$

A $d-$monotone operator is coercive if ρ is (weakly) coercive, that is, when $\rho(x) \to \infty$ as $x \to \infty$, where function ρ is from (4). Uniformly (strongly) monotone operators are necessarily coercive. Proofs of implications in Figure 1 can be found in [11] and [12].

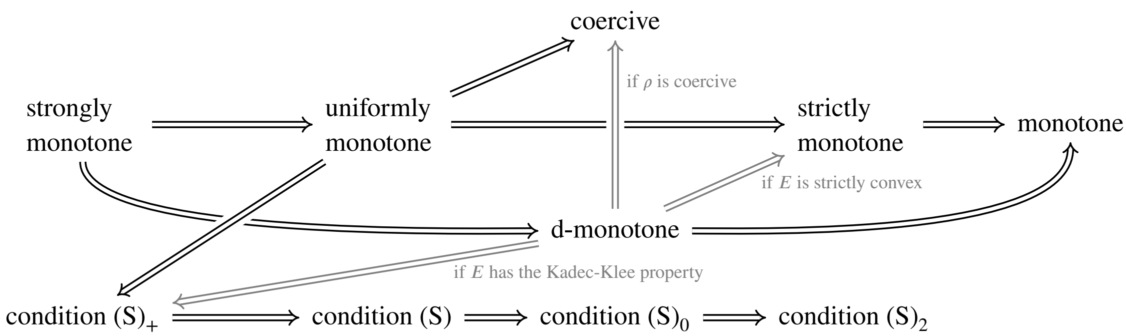

FIGURE 1 Relations between properties of $A : E \to E^*$.

Operator $A : E \to E^*$ is called:

- *demicontinuous* if $u_n \to u_0$ in E implies $A(u_n) \rightharpoonup A(u_0)$ in E^*;
- *hemicontinuous* if for any $u, v, h \in E$ function
$$s \mapsto \langle A(u + sv), h\rangle$$
is continuous on $[0, 1]$;
- *radially continuous* if for any $u, v \in E$ function
$$s \mapsto \langle A(u + sv), v\rangle$$
is continuous on $[0, 1]$;

For any operator each former continuity notion implies the latter, while the converse implications holds, when the operator is monotone.

Operator $A : E \to E^*$ is called *potential*, if there exists a Gâteaux differentiable functional $\mathcal{A} : E \to \mathbb{R}$, called the *potential* of A, such that $\mathcal{A}' = A$. For a radially continuous potential operator $A : E \to E^*$ its potential satisfies that

$$\mathcal{A}(u) = \mathcal{A}(0) + \int_0^1 \langle A(su), u\rangle\, ds \quad \text{for } u \in E.$$

Clearly, the potential (if exists) is determined uniquely up to a constant. Therefore we will assume $\mathcal{A}(0) = 0$. Consequently

$$\mathcal{A}(u) = \int_0^1 \langle A(su), u\rangle\, ds \quad \text{for } u \in E.$$

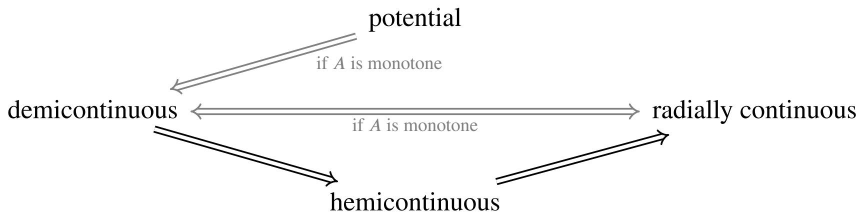

FIGURE 2 Relations between properties of $A:\ E \to E^*$.

When $A:\ E \to E^*$ is potential and monotone then its potential $\mathcal{A}$ is sequentially weakly lower semicontinuous and operator A is demicontinuous. Relation between the above introduced types of continuity are sumarized in Figure 2 .

The very first result in the field of monotone operators utilizes the Banach Contraction Principle in its proof which means that in this special case the dependence on parameters can be easily obtained via Theorem 1.

Theorem 3 (Zarantonello)**.** Assume E is a Hilbert space and $A:\ E \to E$ is Lipschitz continuous, i.e. there is $M > 0$ such that for all $u, v \in E$ we have

$$\|A(u) - A(v)\|_* \leq M\,\|u - v\|\,,$$

and m-strongly monotone with $m < M$. Then for each $h \in E$ equation

$$A(u) = h$$

has exactly one solution. Moreover, A is invertible and $A^{-1}:\ E \to E$ is Lipschitz continuous.

The well known existence result in the theory of monotone operators is

Theorem 4 (Strongly Monotone Principle)**.** Assume that $A:\ E \to E^*$ is radially continuous and strongly monotone. Then operator A is invertible and its inverse $A^{-1}:\ E^* \to E$ is Lipschitz continuous. If additionally operator A is Lipschitz continuous, then A^{-1} is strongly monotone.

In opposite to the Zarantonello result, the proof of the Strongly Monotone Principle is based on the Brouwer Fixed Point Theorem and not on the Banach Contraction Principle and can be found in [12] or [13]. The version of the Browder–Minty Theorem about existence and uniqueness is as follows:

Theorem 5 (Browder–Minty)**.** Assume that $A:\ E \to E^*$ is radially continuous, coercive and strictly monotone. Then for any $f \in E^*$ there exists exactly one solution to equation $A(u) = f$.

One comment is in order to conclude this section. This is about how to understand the Browder–Minty Theorem in the context required for our investigations in this work:

Remark 2. It is known that if $A:\ E \to E^*$ is radially continuous, strictly monotone, coercive and if it satisfies condition (S), then A is invertible and $A^{-1}:\ E^* \to E$ is bounded (on bounded sets), strictly monotone and continuous.

Following Remark 2 it makes sense to consider the solution operator $S:\ E^* \to E$ defined by

$$S(f) = u \iff A(u) = f.$$

With assumptions as in Remark 2 operator S is continuous and this sets path for our next investigations.

2.2 | Main theoretical results

In view of Theorem 1 it is interesting to ask if one can find a counterpart of Theorem 4 dependent on parameters as well. To investigate such a problem we introduce some general assumptions on A, which constitute parameter type counterparts of properties mentioned in Section 2.1.

Recall that Y is a given metric space and a E is real, reflexive and separable Banach space. For operator $A: E\times Y \to E^*$ we consider the following hypotheses:

$$\text{Operator } A(\cdot, y) \text{ is radially continuous for all } y \in Y, \text{ operator } A(u, \cdot) \text{ is continuous for all } u \in E. \tag{C}$$

$$\text{For every } y \in Y \text{ operator } A(\cdot, y) \text{ is strictly monotone.} \tag{M}$$

$$\left.\begin{array}{l}\text{For every } y_0 \in Y \text{ there are an open neighbourhood } V \text{ and an increasing function}\\ \rho_V : [0,\infty) \to [0,\infty) \text{ such that}\\ \qquad \langle A(u,y) - A(v,y), u - v\rangle \geq \big(\rho_V(\|u\|) - \rho_V(\|v\|)\big)(\|u\| - \|v\|)\\ \text{for all } y \in V \text{ and every } u, v \in E.\end{array}\right\} \tag{M$_\mathrm{d}$}$$

$$\left.\begin{array}{l}\text{For every } y_0 \in Y \text{ there are an open neighbourhood } V \text{ and an increasing function}\\ \rho_V : [0,\infty) \to [0,\infty) \text{ such that}\\ \qquad \langle A(u,y) - A(v,y), u - v\rangle \geq \rho_V(\|u - v\|)\, \|u - v\|\\ \text{for all } y \in V \text{ and every } u, v \in E.\end{array}\right\} \tag{M$_\mathrm{u}$}$$

$$\left.\begin{array}{l}\text{There exists a continuous function } m: Y \to (0,\infty) \text{ such that}\\ \qquad \langle A(u,y) - A(v,y), u - v\rangle \geq m(y)\|u - v\|^2\\ \text{for all } u, v \in E \text{ and every } y \in Y.\end{array}\right\} \tag{M$_\mathrm{s}$}$$

$$\left.\begin{array}{l}\text{For every } y_0 \in Y \text{ there are an open neighbourhood } V \text{ and a coercive function } \gamma_V : [0,\infty) \to \mathbb{R}\\ \text{such that}\\ \qquad \langle A(u,y), u\rangle \geq \gamma_V(\|u\|)\|u\|\\ \text{for all } u \in E \text{ and every } y \in V.\end{array}\right\} \tag{K}$$

$$\text{For every } y \in Y \text{ operator } A(\cdot, y) \text{ has a potential } \mathcal{A}(\cdot, y), \text{ which satisfies } \mathcal{A}(0, y) = 0. \tag{P}$$

$$\left.\begin{array}{r} u_n \rightharpoonup u_0 \text{ in } E,\\ y_n \to y_0 \text{ in } Y,\\ A(u_n, y_n) \to A(u_0, y_0) \text{ in } E^*\end{array}\right\} \Rightarrow\ u_n \to u_0 \text{ in } E. \tag{S$_2$}$$

Moreover, we did not introduce a parameter type counterpart of conditions $(S)_+$, (S) and $(S)_0$ since reasonable result can be obtained using the weakest, that is $(S)_2$. In assumption (M_s) we do not need to assume that function m is bounded away from 0.

Remark 3. Figure 3 generalizes Figure 1 into parameter-type setting. To be more precise: Implication (M_d) $\Rightarrow$ (K) holds if for every $y_0 \in Y$ one can find an open neighbourhood V of y_0 and a coercive and increasing function $\rho_V : [0,\infty) \to [0,\infty)$ such that (M_d) holds. Detailed proofs are to be found in Appendix since they follow the proofs of results well known in the literature. With assumptions introduced and explained we may follow with the introduction of our main theoretic results.

Theorem 6. Assume that (C), (M) and (K) hold. Let $y_n \to y_0$ in Y and $f_n \to f_0$ in E^*. Then for every $n \in \mathbb{N}_0$ problem

$$A(u, y_n) = f_n \tag{5}$$

has a unique solution $u_n := S(f_n, y_n) \in E$. Moreover

$$S(f_n, y_n) \rightharpoonup S(f_0, y_0) \quad \text{in } E. \tag{6}$$

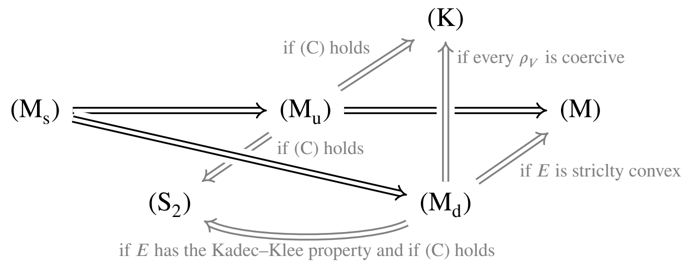

FIGURE 3 Relations between properties of an operator $A : E \times Y \to E^*$.

Proof. Let $f_n \to f_0$ in E^* and $y_n \to y_0$ in Y. From the Browder–Minty Theorem it follows that for each $n \in \mathbb{N}_0$ there exists exactly one u_n which solves (5). Then by (K) there exists a coercive function $\gamma : [0, \infty) \to \mathbb{R}$, for some open neighbourhood of V of y_0, such that for all but a finite number of $n \in \mathbb{N}$ we have

$$\|A(u, y_n)\| \geq \gamma(\|u\|) \quad \text{for all } u \in E.$$

Consequently for all sufficiently large $n \in \mathbb{N}$ we have

$$\|f_n\| \geq \gamma(\|S(f_n, y_n)\|) \quad \text{for all } u \in E.$$

Therefore we have $u_n = S(f_n, y_n) \rightharpoonup u$ for some $u \in E$ (possibly up to subsequence which we do not renumber). Fix $v \in E$ and let $v_t = u - t(u - v)$, $t > 0$. Then the assumption (M) provides

$$\langle A(u_n, y_n) - A(v_t, y_n), u_n - u\rangle + t\langle A(u_n, y_n), u - v\rangle > t\langle A(v_t, y_n), u - v\rangle.$$

By definition of u_n we obtain that $A(u_n, y_n) \to f_0$, $A(v_t, y_n) \to A(v_t, y_0)$ and $u_n \rightharpoonup u$. Therefore

$$\lim_{n\to\infty} \langle A(u_n, y_n) - A(v_t, y_n), u_n - u\rangle = 0.$$

Hence we get that for every $t > 0$

$$\langle f_0, u - v\rangle = \lim_{n\to\infty} \langle A(u_n, y_n), u - v\rangle \geq \lim_{n\to\infty} \langle A(v_t, y_n), u - v\rangle = \langle A(v_t, y_0), u - v\rangle.$$

Letting $t \to 0$ and using assumption (C) again we obtain

$$\langle f_0, u - v\rangle \geq \langle A(u, y_0), u - v\rangle \quad \text{for all } v \in E.$$

Since v was taken arbitrary we see that $u = S(f_0, y_0)$ and consequently by the uniqueness $S(f_n, y_n) \rightharpoonup S(f_0, y_0)$. □

By the proof of Theorem 6 it makes sense to consider *the solution operator* $S : E^* \times Y \to E$ as follows:

$$S(f, y) = u \iff A(u, y) = f \tag{7}$$

provided that assumptions (C), (M) and (K) hold. Then Theorem 6 says that the solution operator is demicontinuous in a sense described by (6). In the potential case, we receive some additional information on the convergence of the minimal value of corresponding Euler action functionals. Indeed, under (P) if we consider equation

$$A(u, y) = f,$$

then the corresponding Euler action functional, i.e. a functional for which the given equation provides a critical point is as follows:

$$\mathcal{A}_f(u, y) = \mathcal{A}(u, y) - \langle f, u\rangle$$

Finding solution to $A(u, y) = f$ thus relies on minimizing $\mathcal{A}_f(\cdot, y)$, which can be performed when functional $\mathcal{A}(\cdot, y)$ is sequentially weakly lowersemicontinuous and coercive for each $y \in Y$. Since a coercive, monotone and radially continuous potential operator ensures such properties for its Euler action functional, we may consider the following result:

Theorem 7. Assume that (C), (M), (K) and (P) hold. Let $y_n \to y_0$ in Y and $f_n \to f_0$ in E^*. Then for operator S given by (7) we have

$$\mathcal{A}(S(f_n, y_n), y_n) = \inf_{v\in E} \left(\mathcal{A}(v, y_n) - \langle f_n, v\rangle\right) \tag{8}$$

for all $n \in \mathbb{N}_0$. Moreover

$$\mathcal{A}\big(S(f_n, y_n), y_n\big) \to \mathcal{A}\big(S(f_0, y_0), y_0\big) \quad \text{as } n \to \infty. \tag{9}$$

Proof. From Theorem 6 for each $n \in \mathbb{N}_0$ there exists exactly one $u_n := S(f_n, y_n)$ for $n \in \mathbb{N}_0$. Relation (8) follows by condition (M) which provides the uniqueness of a critical point. We now show that

$$\mathcal{A}\left(u_n, y_n\right) \to \mathcal{A}\left(u_0, y_0\right) \quad \text{as } n \to \infty$$

We have for $n \in \mathbb{N}$

$$\mathcal{A}\left(u_n, y_n\right) - \mathcal{A}\left(u_0, y_0\right) = \mathcal{A}\left(u_n, y_n\right) - \mathcal{A}\left(u_0, y_n\right) + \mathcal{A}\left(u_0, y_n\right) - \mathcal{A}\left(u_0, y_0\right)$$

Obviously $\mathcal{A}\left(v, y_n\right) \to \mathcal{A}\left(v, y_0\right)$ as $n \to \infty$. Observe that $\langle f_n, u_n\rangle - \langle f_0, v\rangle \to 0$ as $n \to \infty$. By the Gâteaux differentiability of J and since (8) holds we further see that

$$0 \geq \mathcal{A}\left(u_n, y_n\right) - \langle f_n, u_n\rangle - \left(\mathcal{A}\left(u_0, y_n\right) - \langle f_n, u_0\rangle\right) \geq \left\langle A\left(u_0, y_n\right) - f_n, u_n - u_0\right\rangle \to 0 \quad \text{as } n \to \infty.$$

Since $\left(A\left(u_0, y_n\right) - f_n\right) \to \left(A\left(u_0, y_0\right) - f_0\right)$ and $u_n \rightharpoonup u_0$, we see that $\left\langle A\left(u_0, y_n\right) - f_n, u_n - v\right\rangle \to 0$ as $n \to \infty$. Therefore we obtain (9). □

Relations (8) and (9) are not direct consequences of the weak convergence $S(f_n, y_n) \rightharpoonup S(f_0, y_0)$ obtained in Theorem 6. It is illustrated by Example 1, which will be provided later on.

Theorem 8. Assume that (C), (M), (K) and (S_2) hold. Let $y_n \to y_0$ in Y and $f_n \to f_0$ in E^*. Then the solution operator S given by (7) is continuous, i.e.

$$S(f_n, y_n) \to S(f_0, y_0) \text{ in } E. \tag{10}$$

Proof. Let $y_n \to y_0$ and $f_n \to f_0$. Then $S(f_n, y_n) \rightharpoonup S(f_0, y_0)$ by Theorem 6. Therefore we can use the assumption (S_2) to obtain $S(f_n, y_n) \to S(f_0, y_0)$. □

Using Remark 3 and the Theorem 8 we instantly get

Corollary 1. Assume that (C) holds. If additionally either (M_s) or (M_u) is satisfied, then (10) holds. Moreover, if E has the Kadec–Klee property, then (M_d) provides (10) as well.

The following theorem serves as some extension of Theorem 1 in a Hilbert space setting.

Theorem 9. Assume that A satisfies (C) and that E is a Hilbert space. If there exists a continuous function $L : Y \to (0, 1)$ such that

$$\langle A(u, y) - A(v, y), u - v\rangle \leq L(y)\|u - v\|^2 \tag{11}$$

for all $u, v \in E$, then for every $y \in Y$ the map $u \mapsto A(u, y)$ has a unique fixed point $\Phi(y)$. Moreover, the function $y \mapsto \Phi(y)$ is continuous.

Proof. Define an auxiliary operator $T : E \times Y \to E^*$ by the formula

$$T(u, y) = u - A(u, y).$$

By Corollary 1 we can apply Theorem 8 to operator T in order to obtain the assertion. □

Condition (11) is called *relaxed monotinicity condition* or *one-sided Lipschitz condition*. It is weaker than the Lipschitz condition since whenever

$$\|A(u, y) - A(v, y)\|_* \leq L(y)\|u - v\|$$

holds, we also get

$$\langle A(u, y) - A(v, y), u - v\rangle \leq \|A(u, y) - A(v, y)\|_* \|u - v\| \leq L(y)\|u - v\|^2.$$

Notice that one can use Theorems 6 and 8 in order to consider some pointwise convergent sequences of radially continuous operators (T_n), that is $T_n : E \to E^*$ is radially continuous for all $n = 0, 1, 2, \dots$ and $T_n(u) \to T_0(u)$ when $n \to \infty$ for all $u \in E$. It is sufficient to take $Y = \mathbb{N}_0 = \{0, 1, 2, \dots\}$ and equip Y with a metric

$$d(n, m) = \left|\frac{n}{1 + n^2} - \frac{m}{1 + m^2}\right|. \tag{12}$$

Then a mapping $A : Y \times E \to E^*$ given by $A(u, n) = T_n(u)$ clearly satisfies assumption (C). The following example shows that although assumption (M_s) ensures (M), (K) and (S_2), the continuity of function m is nevertheless crucial.

Example 1. Consider $\psi_n : \mathbb{R} \to \mathbb{R}$ given by

$$\psi_n(x) := \begin{cases} \frac{x}{n} & \text{if } |x| \le 1, \\ \frac{2n-1}{n}x - \frac{2n-2}{n} & \text{if } 1 < |x| \le 2, \\ x & \text{if } 2 < |x| \end{cases}$$

and let $K : \ell^2 \times \mathbb{N}_0 \to \ell^2$, where $\mathbb{N}_0$ is equipped with a metric d defined by (12), be given by

$$K(u,n)(i) = \begin{cases} u(i) & \text{for } i \ne n, \\ \psi_n(u(i)) & \text{for } i = n. \end{cases}$$

for $n \in \mathbb{N}$ and by $K(u,0) = u$ for all $u \in \ell^2$. Notice that for every fixed $u \in \ell^2$ we have $u(n) < 1$ for sufficiently large n. Therefore, for sufficiently large n we have

$$K(u,n) - K(u,0) = \frac{n-1}{n}u(n),$$

which provides $K(u,n) \to K(u,0)$ since $u(n) \to 0$. The continuity of $K(\cdot,n)$, $n \in \mathbb{N}$, is clear. Hence (C) holds. Moreover, for every $n \in \mathbb{N}$ we have

$$\langle K(u,n) - K(v,n), u - v\rangle \ge \frac{1}{n}\|u - v\|^2 \quad \text{for all } u, v \in \ell^2.$$

and

$$\langle K(u,0) - K(v,0), u - v\rangle \ge \|u - v\|^2 \quad \text{for all } u, v \in \ell^2,$$

which gives (M). Since $\psi_n(x)x \ge x^2 - 8$ for all $n \in \mathbb{N}$ and every $x \in \mathbb{R}$, then for every $u \in \ell^2$ there is

$$\langle K(u,n), u\rangle = \sum_{i \ne n} |u(i)|^2 + \psi_n(u(n))u(n) \ge \sum_{i=1}^{\infty} |u(i)|^2 - 8 = \|u\|^2 - 8.$$

Hence assumption (K) is satisfied. Denote

$$e_n(i) = \begin{cases} 0 & \text{for } i \ne n, \\ 1 & \text{for } i = n. \end{cases}$$

Since $K(e_n,n) = \frac{e_n}{n}$, then we obtain $\frac{e_n}{n} \to 0$. However $e_n \not\to 0$ and $K(0,0) = 0$. Notice (P) holds with

$$\mathcal{K}(u,n) = \frac{1}{2}\sum_{i \ne n} |u(i)|^2 + \int_0^{u(n)} \psi_n(s)ds.$$

Finally notice that $e_{n+1} \rightharpoonup 0$ as $n \to \infty$, while $\mathcal{K}(e_{n+1},n) = \frac{1}{2} \not\to 0 = \mathcal{K}(0,0)$.

3 | APPLICATIONS TO BOUNDARY VALUE PROBLEMS

Throughout this section we assume

$$\Omega \subset \mathbb{R}^N,\ N \in \mathbb{N}, \text{ is open and bounded set with a Lipschitz boundary,} \tag{Ω}$$

$$\Sigma \subset \mathbb{R} \text{ is a closed set.} \tag{Σ}$$

For every $p \in [1,\infty) \cup \{\infty\}$ we denote

$$L^p(\Omega;\Sigma) := \{w \in L^p(\Omega) : w(x) \in \Sigma \text{ for a.e. } x \in \Omega\}$$

and equip $L^p(\Omega;\Sigma)$ with a topology inherited from $L^p(\Omega)$. We introduce *Sobolev spaces* following[14]. Denote by $W^{1,p}(\Omega)$, $1 < p < \infty$, the space of all functions from $L^p(\Omega)$, whose all first order weak derivatives belong to $L^p(\Omega)$. We equip $W^{1,p}(\Omega)$ with a norm

$$\|u\|_{W^{1,p}} := \left(\int_\Omega |u(x)|^p dx + \int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}}.$$

Let $C_0^\infty(\Omega)$ denotes a space of all smooth functions with a compact support in Ω. Space $W_0^{1,p}(\Omega)$ is a closure of $C_0^\infty(\Omega)$ in $\|\cdot\|_{W^{1,p}}$-norm. We consider

$$\|u\|_{W_0^{1,p}} := \left(\int_\Omega |\nabla u(x)|^p dx\right)^{\frac{1}{p}}.$$

Dual space of $W_0^{1,p}(\Omega)$ is

$$W^{-1,p'}(\Omega) := \left(W_0^{1,p}(\Omega)\right)^*, \quad \text{where } \frac{1}{p} + \frac{1}{p'} = 1.$$

We take $H^1(\Omega) := W^{1,2}(\Omega)$, $H_0^1(\Omega) := W_0^{1,2}(\Omega)$, $H^{-1}(\Omega) := W^{-1,2}(\Omega)$ and denote by

$$\lambda_1(\Omega) = \inf_{u\in H_0^1(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}$$

the Poincaré constant. It is well known that $\lambda_1(\Omega) > 0$ if (Ω) holds.

3.1 | Perturbated p-Laplacian operators

Elliptic differential operators in divergence form find many applications in modelling various of physical phenomenons, see for instance [15]. Operators of type

$$u \mapsto \operatorname{div}(a\nabla u)$$

appear in equations of heat or electric conduction, where u stand for, respectively, temperature or electric potential. Quantity a characterises some material. In general it is not constant and can depend on influencing factors. More advanced phenomenon can be found in nonlinear flows in channels and ditches, and filtration of fluids through porous media. The above consideration lead to introduction of the p-Laplace operator, namely

$$u \mapsto \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right). \tag{13}$$

The work [16] provides a detailed and precise survey of the p-laplacian and its properties. Moreover some applications in model of the thermistor are given in [15]. Variational and monotonicity methods are important for the analysis of various problems ralated to the field of mathematical physis, like for example the following [17,18,19] which also deal with the p-Laplacian and double phase boundary value problems.

In this paper we consider perturbation (depending on function parameter) of operator (13) considered in the weak form. For $p \geq 2$ and $\varphi : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ we define $D_{p,\varphi} : W_0^{1,p}(\Omega) \times L^\infty(\Omega) \to W^{-1,p'}(\Omega)$ by

$$\langle D_{p,\varphi}(u,y), v\rangle = \int_\Omega \varphi\left(x, y(x), |\nabla u(x)|^{p-1}\right)|\nabla u(x)|^{p-2}\nabla u(x)\cdot\nabla v(x)dx.$$

Consider the following hypotheses on φ, which will lead to the well posedness, continuity, coercivity and monotonicity properties of the operator.

(φ_C):

$\varphi : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ is *a Carathéodory function*, that is:

- $\varphi(\cdot, y, r)$ is measurable for all $y \in \mathbb{R}$ and $r \geq 0$;
- $\varphi(x, \cdot, r)$ and $\varphi(x, y, \cdot)$ are continuous for a.e. $x \in \Omega$ and every $y \in \mathbb{R}$, $r \geq 0$.

Moreover, there exists a constant $M > 0$ such that

$$|\varphi(x, y, r)| \leq M$$

for a.e. $x \in \Omega$, all $y \in \mathbb{R}$ and $r \geq 0$.

$$\left.\begin{array}{l}\text{There exist a constant } \gamma > 0 \text{ such that}\\ \qquad\qquad \varphi(x,y,r) \geq \gamma \\ \text{for a.e. } x \in \Omega, \text{ all } y \in \mathbb{R} \text{ and } r \geq 0.\end{array}\right\} \tag{φ_{K}}$$

$$\left.\begin{array}{l}\text{For a.e. } x \in \Omega, \text{ every } y \in \mathbb{R} \text{ and all } r > s \geq 0. \text{ there is}\\ \qquad\qquad \varphi(x,y,r)\,r - \varphi(x,y,s)\,s > 0\end{array}\right\} \tag{φ_{M}}$$

$$\left.\begin{array}{l}\text{There exist a constant } \gamma > 0 \text{ such that}\\ \qquad\qquad \varphi(x,y,r)\,r - \varphi(x,y,s)\,s \geq \gamma\,(r-s) \\ \text{for a.e. } x \in \Omega, \text{ all } y \in \mathbb{R} \text{ and } r \geq s \geq 0.\end{array}\right\} \tag{φ_{S}}$$

Example 2. Let us consider functions $\varphi_1, \varphi_2 : \Omega \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$ given by

$$\varphi_1(x,y,r) = \begin{cases} 1+y^2 & \text{if } r \in [0,1) \cup [2,\infty), \\ r-2+\frac{2}{r}+y^2 & \text{if } r \in [1,2), \end{cases}$$
$$\varphi_2(x,y,r) = |\arctan y| + 1.$$

Then both, φ_1 and φ_2, satisfy (φ_{C}), (φ_{K}) and (φ_{M}). Moreover, φ_2 satisfies (φ_{S}), while φ_1 does not.

Lemma 1. If function φ satisfies assumption

- (φ_{C}), then operator $D_{p,\varphi}$ satisfies condition (C);
- (φ_{K}), then operator $D_{p,\varphi}$ satisfies condition (K);
- (φ_{M}), then operator $D_{p,\varphi}$ satisfies condition (M);
- (φ_{S}) and if additionally
  - $p = 2$, then operator $D_{p,\varphi}$ satisfies condition ($\mathrm{M_s}$) with
    $$m(y) = \gamma,$$
    for every $y \in L^\infty(\Omega)$;
  - $p > 2$, then operator $D_{p,\varphi}$ satisfies condition ($\mathrm{M_d}$) with function
    $$\rho_V(r) = \gamma r^{p-1},$$
    where for every $y_0 \in L^\infty(\Omega)$ we take $V = L^\infty(\Omega)$.

Figure 4 summarizes briefly the above mentioned relations which are proved in the Appendix.

$$\begin{tikzcd}
\text{Function } \varphi & (\varphi_{\mathrm{C}}) \arrow[d, Rightarrow] & (\varphi_{\mathrm{K}}) \arrow[d, Rightarrow] & (\varphi_{\mathrm{M}}) \arrow[d, Rightarrow] & (\varphi_{\mathrm{S}}) \arrow[d, Rightarrow, "\text{if } p=2"'] \arrow[dr, Rightarrow, "\text{if } p>2"] & \\
\text{Operator } D_{p,\varphi} & (\mathrm{C}) & (\mathrm{K}) & (\mathrm{M}) & (\mathrm{M_s}) & (\mathrm{M_d})
\end{tikzcd}$$

FIGURE 4 Relations between φ and $D_{p,\varphi}$.

Notice that $D_{p,\varphi}$ satisfies (P) with $\mathcal{D}_{p,\varphi} : W_0^{1,p}(\Omega) \times L^\infty(\Omega) \to \mathbb{R}$ given by

$$\mathcal{D}_{p,\varphi}(u,y) = \int_\Omega \int_0^{|\nabla u(x)|} \varphi\big(x, s^{p-1}, y(x)\big) s^{p-1}\, ds\, dx.$$

3.2 | Nemytskii operator

Assume the following

$$\left.\begin{array}{l} f:\ \Omega\times\mathbb{R}\times\Sigma\to\mathbb{R} \text{ is } \textit{a Carathéodory function}: \\ \bullet\ f(\cdot,u,w) \text{ is measurable for all } u\in\mathbb{R} \text{ and } w\in\Sigma; \\ \bullet\ f(x,\cdot,w) \text{ and } f(x,u,\cdot) \text{ are continuous for a.e. } x\in\Omega \text{ and every } u\in\mathbb{R} \text{ and } w\in\Sigma. \end{array}\right\} \qquad (f_C)$$

(f_p):

- If $N < p$, then there exists $\alpha \in L^1(\Omega;[0,\infty))$ and a continuous function $\gamma:\ [0,\infty)\times\Sigma\to[0,\infty)$ such that
$$|f(x,u,w)| \le \alpha(x)+\gamma(|u|,w)$$
for a.e. $x\in\Omega$ and every $u\in\mathbb{R}$, $w\in\Sigma$,
- if $N=p$, then there exists $c>1$, $\beta>0$, $a\in L^\gamma(\Omega;[0,\infty))$ and continuous $g:\ \Sigma\to[0,\infty)$ such that
$$|f(x,u,w)| \le \alpha(x)+\beta|u|^c+g(w)$$
for a.e. $x\in\Omega$ and every $u\in\mathbb{R}$, $w\in\Sigma$,
- if $N>p$, then there exist $a\in L^{Np/(N+p)}(\Omega;[0,\infty))$, $\beta>0$ and a continuous function $g:\ \Sigma\to[0,\infty)$ such that
$$|f(x,u,w)| \le \alpha(x)+\beta|u|^{(N+p)/(N-p)}+g(w)$$
for a.e. $x\in\Omega$ and every $u\in\mathbb{R}$, $w\in\Sigma$,

We define *the Nemytskii operator* $N_{p,f}:\ W_0^{1,p}(\Omega)\times L^\infty(\Omega;\Sigma)\to W^{-1,p'}(\Omega)$ associated with function f by

$$\langle N_{p,f}(u,w),v\rangle=\int_\Omega f\big(x,u(x),w(x)\big)v(x)dx.$$

The following well known result relates the Krasnosielki Theorem with the Rellich–Kondrachov Theorem.

Proposition 1. Assume that f satisfies (f_C) and (f_p). Then the Nemytskii operator is well defined and continuous. Therefore, in particular, it satisfies (C).

Operator $N_{p,f}$ satisfies (P), i.e. it is potential with the potential with $\mathcal{N}_{p,f}:\ W_0^{1,p}(\Omega)\times L^\infty(\Omega;\Sigma)\to\mathbb{R}$ given by

$$\mathcal{N}_{p,f}(u,w)=\int_\Omega\int_0^{u(x)} f(x,s,w(x))dsdx.$$

3.3 | A boundary value problem with the perturbated p-Laplacian

Let $p\ge 2$. We study parameter dependence of solutions to the following boundary value problem

$$\begin{cases} -\operatorname{div}\Big(\varphi\big(x,y_n(x),|\nabla u(x)|^{p-1}\big)|\nabla u(x)|^{p-2}\nabla u(x)\Big)=f(x,u(x),w_n(x))+\boldsymbol{a}_n(x)\cdot\nabla u(x)+g_n(x), \\ u|_{\partial\Omega}=0. \end{cases} \qquad (14)$$

We consider the following hypotheses:

$$\left.\begin{array}{l} g_n\to g_0 \text{ in } L^p(\Omega), \\ y_n\to y_0 \text{ in } L^\infty(\Omega), \\ w_n\to w_0 \text{ in } L^\infty(\Omega;\Sigma), \\ \boldsymbol{a}_n\to\boldsymbol{a}_0 \text{ in } L^\infty(\Omega;\mathbb{R}^N). \end{array}\right\} \qquad (A)$$

There exists a constant $m_r \in \mathbb{R}$ such that
$$\big(f(x,u,w) - f(x,v,w)\big)(u-v) \le m_r|u-v|^2 \tag{f_{R}}$$
for all $x \in \Omega$ and every $u, v \in \mathbb{R}$, $w \in \Sigma$.

We define functionals $\mathcal{E}_{p,n} : W_0^{1,p}(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega;\Sigma) \times L^p(\Omega) \to \mathbb{R}$, $n \in \mathbb{N}_0$, by
$$\mathcal{E}_{p,n}(u,y,w,g) = \int_\Omega \int_0^{|\nabla u(x)|} \varphi\big(x, s^{p-1}, y(x)\big) s^{p-1} ds\,dx - \int_\Omega \int_0^{u(x)} f(x,s,w(x))ds\,dx - \int_\Omega g_n(x)u(x)dx. \tag{15}$$

Let us recall that $u \in W_0^{1,p}(\Omega)$ is *a weak solution* to (14) if for every $v \in W_0^{1,p}(\Omega)$ we have
$$\langle D_{p,\varphi}(u,y_n), v\rangle = \int_\Omega f\big(x,u(x),y_n(x)\big)v(x)dx + \int_\Omega \big(\boldsymbol{a}_n(x)\cdot\nabla u(x) + g_n(x)\big)v(x)dx.$$

Example 3. Let functions $f_1, f_2, f_3 : \Omega \times \mathbb{R} \times \Sigma \to \mathbb{R}$ be defined by
$$\begin{aligned} f_1(x,u,w) &:= -u^k e^{-w^2} + |w| \arctan u,\\ f_2(x,u,w) &:= we^{-u} + u\sin w,\\ f_3(x,u,w) &:= w|x|^2\left(u - u^k\right), \end{aligned}$$
where k is odd natural number. Then f_1 satisfies (f_{R}) if Σ is bounded, while f_2 satisfies (f_{R}) provided that $\Sigma \subset [0,\infty)$. Moreover, if both hold, that is if Σ is bounded set and if $\Sigma \subset [0,\infty)$, then assumptions (f_{R}) holds for f_3.

3.3.1 | Results with $p = 2$

Recall that *the essential range* of measurable function $\boldsymbol{g} : \Omega \to \mathbb{R}^N$ is given by
$$\operatorname{ess.ran} \boldsymbol{g} = \left\{\xi \in \mathbb{R}^N : \forall\varepsilon > 0 \quad \mu\big(\{x : |\boldsymbol{g}(x) - \xi| < \varepsilon\}\big) > 0\right\},$$
where μ is an N-dimensional Lebesgue measure on $\mathbb{R}^N$. Notice that for every a.e. constant function $\boldsymbol{g}$ its essential range is a singleton.

Theorem 10. Let $p = 2$ and assume that (Ω), (Σ) and (A) hold. Take φ satisfying (φ_{C}) and (φ_{S}) and let f satisfy (f_{C}), (f_p) and (f_{R}). Define
$$c := \sup_{n\in\mathbb{N}_0} \big(\operatorname{diam}(\operatorname{ess.ran} \boldsymbol{a}_n)\big).$$
If moreover
$$\gamma\lambda_1(\Omega) > m_r + \tfrac{\sqrt{\lambda_1(\Omega)}}{2}c,$$
then $u_n \to u_0$ in $H_0^1(\Omega)$, where u_n is the unique weak solution to (14) for every $n \in \mathbb{N}_0$.

Proof. By Lemma 1 operator $D_{2,\varphi}$ satisfies $(\mathrm{M_s})$ with $m \equiv \gamma$. By condition (f_{R}) we also have
$$\langle N_{2,f}(u,w) - N_{2,f}(v,w), u - v\rangle \le m_r \int_\Omega |u(x) - v(x)|^2 dx \le \tfrac{m_r}{\lambda_1(\Omega)} \int_\Omega |\nabla u(x) - \nabla v(x)|^2 dx$$
For every $n \in \mathbb{N}_0$ we take $\overline{\boldsymbol{a}_n} \in \mathbb{R}^N$ such that
$$\operatorname{ess.ran} \boldsymbol{a}_n \subset B\left(\overline{\boldsymbol{a}_n}, \tfrac{c}{2}\right).$$
Therefore for every $n \in \mathbb{N}_0$ we have
$$\operatorname*{ess.sup}_{x\in\Omega} |\boldsymbol{a}_n(x) - \overline{\boldsymbol{a}_n}| \le \tfrac{c}{2}.$$

Moreover for every $n \in \mathbb{N}_0$ we have

$$\begin{aligned}\int_\Omega \boldsymbol{a}_n(x)\cdot\nabla u(x)u(x)dx &= \int_\Omega \left(\boldsymbol{a}_n(x)-\overline{\boldsymbol{a}_n}\right)\cdot\nabla u(x)u(x)dx + \int_\Omega \overline{\boldsymbol{a}_n}\cdot\nabla u(x)u(x)dx \\ &\le \tfrac{c}{2}\left(\int_\Omega |\nabla u(x)|^2dx\right)^{\frac12}\left(\int_\Omega |u(x)|^2dx\right)^{\frac12} + \int_\Omega \overline{\boldsymbol{a}_n}\cdot\nabla u(x)u(x)dx \\ &\le \tfrac{c}{2\sqrt{\lambda_1(\Omega)}}\int_\Omega |\nabla u(x)|^2dx + \int_\Omega \overline{\boldsymbol{a}_n}\cdot\nabla u(x)u(x)dx,\end{aligned}$$

For every $n \in \mathbb{N}_0$ and $u \in H_0^1(\Omega)$ it holds

$$\begin{aligned}\int_\Omega \overline{\boldsymbol{a}_n}\cdot\nabla u(x)u(x)dx &= \sum_{i=1}^N(\overline{\boldsymbol{a}_n})_i\int_\Omega \frac{\partial u}{\partial x_i}(x)u(x)dx = -\sum_{i=1}^N(\overline{\boldsymbol{a}_n})_i\int_\Omega u(x)\frac{\partial u}{\partial x_i}(x)dx \\ &= -\int_\Omega \overline{\boldsymbol{a}_n}\cdot\nabla u(x)u(x)dx.\end{aligned} \tag{16}$$

Then for all $n \in \mathbb{N}_0$ and every $u \in H_0^1(\Omega)$ we have

$$\int_\Omega \overline{\boldsymbol{a}_n}\cdot\nabla u(x)u(x)dx = 0$$

and hence

$$\int_\Omega \boldsymbol{a}_n(x)\cdot\nabla u(x)u(x)dx \le \tfrac{c}{2\sqrt{\lambda_1(\Omega)}}\int_\Omega |\nabla u(x)|^2dx.$$

Take $A:\ H_0^1(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega;\Sigma)\times\mathbb{N}_0\times L^2(\Omega)\to H^{-1}(\Omega)$ given by

$$A(u,y,w,n,g) = \langle D_{2,\varphi}(u,y),v\rangle - \langle N_{2,f}(u,w),v\rangle - \langle \boldsymbol{a}\cdot\nabla u + g, v\rangle$$

for every $u,v\in H_0^1(\Omega),\ y\in L^\infty(\Omega),\ w\in L^\infty(\Omega;\Sigma),\ n\in\mathbb{N}_0$ and $g\in L^2(\Omega)$ we get

$$\begin{aligned}\langle A(u,y,w,n,g)-A(v,y,w,n,g),u-v\rangle &\ge \left(\gamma-\tfrac{m_r}{\lambda_1(\Omega)}-\tfrac{c}{2\sqrt{\lambda_1(\Omega)}}\right)\int_\Omega |\nabla u(x)-\nabla v(x)|^2dx \\ &= \frac{\gamma\lambda_1(\Omega)-m_r-\frac{c}{2}\sqrt{\lambda_1(\Omega)}}{\lambda_1(\Omega)}\int_\Omega |\nabla u(x)-\nabla v(x)|^2dx.\end{aligned}$$

Since $\mathbb{N}_0$ is equipped with a metric (12), we see that ($\mathrm{M_s}$) holds for the above defined operator A. Moreover by (φ_C), (f_C), Lemma 1 and Proposition 1 it follows that A satisfies condition (C). Hence we can apply Theorem 8 in order to get the assertion. □

3.3.2 | Results with $p > 2$

In this subsection we consider the case of the Dirichlet problem governed by the p-Laplacian. There are some differences when compared to the case of $p = 2$ as far as functions a_n are concerned. This is somehow related to the fact that for the p-Laplacian there are no known relaxed monotonicty type conditions.

Theorem 11. Assume that (Ω), (Σ) and (A) hold. Take φ satisfying (φ_C) and (φ_S) and let f satisfy (f_C), (f_p) and (f_R). If

$$m_r \le 0 \quad \text{and if functions } \boldsymbol{a}_n,\ n\in\mathbb{N}_0, \text{ are constant (a.e.),}$$

then we have $u_n \to u_0$ in $W_0^{1,p}(\Omega)$, where u_n is the unique weak solution to (14) for $n \in \mathbb{N}_0$.

Proof. Define $A:\ W_0^{1,p}(\Omega)\times L^\infty(\Omega)\times L^\infty(\Omega;\Sigma)\times\mathbb{N}_0\times L^p(\Omega)\to W^{-1,p'}(\Omega)$ by

$$A(u,y,w,n,g) = \langle D_{p,\varphi}(u,y),v\rangle - \langle N_{p,f}(u,w),v\rangle - \langle \boldsymbol{a}\cdot\nabla u + g, v\rangle.$$

for every $u, v \in W_0^{1,p}(\Omega),\ y \in L^\infty(\Omega),\ w \in L^\infty(\Omega;\Sigma),\ n \in \mathbb{N}_0$ and $g \in L^p(\Omega)$. Since functions $\boldsymbol{a}_n$ are (a.e.) we can use (16) to obtain

$$\int_\Omega \boldsymbol{a}_n(x) \cdot \nabla u(x) u(x) dx = 0 \quad \text{for all } u \in W_0^{1,p}(\Omega).$$

Moreover assumptions (f_{R}) and $m_r \leq 0$ provide

$$\langle N_{p,f}(u,w) - N_{p,f}(v,w), u - v\rangle \leq 0$$

for all $u, v \in W_0^{1,p}(\Omega)$ and $w \in L^\infty(\Omega;\Sigma)$. Therefore, by (φ_{S}) and Lemma 1, we have

$$\begin{aligned}\langle A(u,y,w,n,g) - A(v,y,w,n,g), u - v\rangle &\geq \langle D_{p,\varphi}(u,y) - D_{p,\varphi}(v,y), u - v\rangle \\ &\geq \left(\|u\|_{W_0^{1,p}} - \|v\|_{W_0^{1,p}}\right)\left(\|u\|^{p-1}_{W_0^{1,p}} - \|v\|^{p-1}_{W_0^{1,p}}\right)\end{aligned}$$

for all $u, v \in W_0^{1,p}(\Omega)$ and every $y \in L^\infty(\Omega)$, $w \in L^\infty(\Omega;\Sigma)$, $n \in \mathbb{N}_0$ and $g \in L^p(\Omega)$. Hence A satisfies $(\mathrm{M_d})$. From (φ_{C}), (f_{C}), Lemma 1 and Proposition 1 it follows that (C) holds for A. Therefore we can use Corollary 1 to get the assertion since $W_0^{1,p}(\Omega)$ has the Kadec–Klee property as a uniformly convex space. □

The potential case of (14) takes place when $\boldsymbol{a}_n = 0$ for $n \in \mathbb{N}_0$. Then we have what follows:

Theorem 12. Assume that (Ω), (Σ) and (A) hold. Take φ satisfying (φ_{C}), (φ_{K}) and (φ_{M}). Let f satisfy (f_{C}), (f_p) and (f_{R}). If

$$m_r \leq 0 \quad \text{and} \quad \boldsymbol{a}_n = 0 \text{ for every } n \in \mathbb{N}_0,$$

then we have $u_n \rightharpoonup u_0$ in $W_0^{1,p}(\Omega)$, where u_n is the unique weak solution to (14) for $n \in \mathbb{N}_0$. Moreover,

$$\mathcal{E}_{p,n}(u_n) \to \mathcal{E}_{p,0}(u_0) \quad \text{and} \quad \mathcal{E}_{p,m}(u_m) = \inf_{v \in W_0^{1,p}(\Omega)} \mathcal{E}_{p,m}(v) \text{ for every } m \in \mathbb{N}_0,$$

where $\mathcal{E}_{p,n}$ is given by (15).

Proof. Analogously as in the proof of Theorem 11 we can show that operator $A:\ W_0^{1,p}(\Omega) \times L^\infty(\Omega) \times L^\infty(\Omega;\Sigma) \times L^p(\Omega) \to W^{-1,p'}(\Omega)$ by

$$A(u,y,w,g) = \langle D_{p,\varphi}(u,y), v\rangle - \langle N_{p,f}(u,w), v\rangle - \langle g, v\rangle.$$

for every $u, v \in W_0^{1,p}(\Omega),\ y \in L^\infty(\Omega),\ w \in L^\infty(\Omega;\Sigma)$ and $g \in L^p(\Omega)$ satisfies (C), (K) and (M). Moreover A satisfies (P) with

$$\mathcal{A}(u, y_n, w_n, g_n) = \mathcal{E}_{p,n}(u).$$

Therefore, the assertion follows from Theorem 7. □

4 | FINAL COMMENTS AND REMARKS

When we consult the proof of the Browder–Minty Theorem we note that the solution of a given nonlinear equation is approximated by a weakly convergent (sub)sequence of Galerkin approximations. The entire sequence converges when uniqueness is involved, while the strong convergence is a consequence the usage of condition (S) or some other related. We see that this observation relates as well to our parametric results. Indeed, we obtain the boundedness of the sequence of solutions corresponding to a convergent sequence of parameters due to the uniform coercivity, while its convergence is proved via monotonicity, some type of continuity and condition $(\mathrm{S})_2$. This observation inspired us to consider some general framework concerning the dependence on parameters as is provided above.

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□

APPENDIX

Proof of Theorem 1. Due to the Banach Contraction Principle, we need to investigate only the continuity of φ. Take $y, y_0 \in Y$ and observe that

$$\begin{aligned} d\left(\Phi(y), \Phi(y_0)\right) &= d\left(T(\Phi(y), y), T(\Phi(y_0), y_0)\right) \leq d\left(T(\Phi(y), y), T(\Phi(y_0), y\right) + d\left(T(\Phi(y_0), y), T(\Phi(y_0), y_0\right) \\ &\leq L(y) d\left(\Phi(y), \Phi(y_0)\right) + d\left(T(\Phi(y_0), y), T(\Phi(y_0), y_0)\right), \end{aligned}$$

which implies that

$$d\left(\Phi(y), \Phi(y_0)\right) \leq \frac{1}{1 - L(y)} d\left(T(\Phi(y_0), y), T(\Phi(y_0), y_0)\right).$$

As $y \to y_0$ in Y, we see by the continuity of T and by $L(y) \to L\left(y_0\right) \in (0,1)$ that $d\left(\Phi(y), \Phi(y_0)\right) \to 0$. □

Proof of Remark 3.

- To show that $(\mathrm{M_s})$ implies $(\mathrm{M_u})$ we fix $y_0 \in Y$ and consider $0 < \varepsilon < m(y_0)$. Taking $V = \{y \in Y : m(y_0) - \varepsilon < m(y)\}$ and $\rho_V(r) = (m(y_0) - \varepsilon) r$ we get $(\mathrm{M_u})$.
- Implication $(\mathrm{M_u}) \Rightarrow (\mathrm{M})$ follows since $\rho_V(0) = 0$ and $\rho_V(r) > 0$ for $r > 0$.
- We show that $(\mathrm{M_d}) \Rightarrow (\mathrm{K})$. Taking $v = 0$ in $(\mathrm{M_d})$ we have
$$\langle A(u, y), u\rangle \geq \left(\rho_V(\|u\|) - \rho_V(0)\right)\|u\| \quad \text{for all } u \in E.$$
Therefore (K) holds provided that every ρ_V is a coercive function.
- We show that (C) and $(\mathrm{M_u})$ implies (K). Fix y_0 and take ρ_V from $(\mathrm{M_u})$. By direct calculations we get
$$\langle A(u, y), u\rangle \geq \|u\|\left(\rho_V(\|u\|) - \|A(0, y)\|_*\right) \quad \text{for all } u \in E$$
Using (C) we can find another neighbourhood $U \subset V$ of y_0 such that $\|A(0, y)\| \leq M < \infty$ for $y \in U$. Hence (K) holds with $\gamma_U(r) = r(\rho_V(r) - M)$ for $y \in U$ and $r \in \mathbb{R}$.
- Implication $(\mathrm{M_s}) \Rightarrow (\mathrm{M_d})$ follows from inequality $\|u - v\| \geq \Big|\|u\| - \|v\|\Big|$.
- Notice that assumption $(\mathrm{M_d})$ provides
$$\langle A(u, y) - A(v, y), u - v\rangle \geq 0 \quad \text{for all } u, v \in E$$
Moreover for any distinct $u, v \in E$ we have
$$\begin{aligned} 0 =& \langle A(u, y) - A(v, y), u - v\rangle \\ =& 2\left\langle A(u, y) - A\left(\tfrac{u+v}{2}, y\right), u - \tfrac{u+v}{2}\right\rangle + 2\left\langle A\left(\tfrac{u+v}{2}, y\right) - A(v, y), \tfrac{u+v}{2} - v\right\rangle \\ \geq& 2\left(\rho(\|u\|) - \rho\left(\left\|\tfrac{u+v}{2}\right\|\right)\right)\left(\|u\| - \left\|\tfrac{u+v}{2}\right\|\right) + 2\left(\rho\left(\left\|\tfrac{u+v}{2}\right\|\right) - \rho(\|v\|)\right)\left(\left\|\tfrac{u+v}{2}\right\| - \|v\|\right) \end{aligned}$$
and hence $\|u\| = \|v\| = \left\|\frac{u+v}{2}\right\|$. Therefore, if E is strictly convex, then $u = v$ and (M) holds.
- We show that (C) and $(\mathrm{M_u})$ implies $(\mathrm{S_2})$. Take $u_n \rightharpoonup u_0$ in E, $y_n \to y_0$ in Y and assume that $A(u_n, y_n) \to A(u_0, y_0)$ in E^*. Let V be an open neighbourhood of y_0 chosen from condition $(\mathrm{M_u})$. Then $y_n \in V$ for sufficiently large n. Hence
$$\begin{aligned} \langle A(u_n, y_n) - A(u_0, y_0), u_n - u_0\rangle =& \langle A(u_n, y_n) - A(u_0, y_n), u_n - u_0\rangle + \langle A(u_0, y_n) - A(u_0, y_0), u_n - u_0\rangle \\ \geq& \rho_V(\|u_n - u_0\|)\|u_n - u_0\| + \langle A(u_0, y_n) - A(u_0, y_0), u_n - u_0\rangle. \end{aligned}$$
By (C) and $(\mathrm{M_u})$ we get $u_n \to u_0$.
- Finally we show (C) and $(\mathrm{M_d})$ implies $(\mathrm{S_2})$. Take $u_n \rightharpoonup u_0$ in E, $y_n \to y_0$ in Y and assume that $A(u_n, y_n) \to A(u_0, y_0)$ in E^*. Using calculations analogous to the previous one we obtain $\|u_n\| \to \|u_0\|$ as $n \to \infty$. The Kadec–Klee property gives $u_n \to u_0$. □

Proof of Lemma 1. Note that in what follows we write sometimes u and y instead of $u(x)$ and $y(x)$ under the integral sign in order to shorten the notation.

- It follows by the Krasnoselskii Theorem on the continuity of the Niemytskii operator that (φ_{C}) implies (C).
- Assume that the function φ satisfies the condition (φ_{K}). Then for any $u \in W_0^{1,p}(\Omega)$ and $y \in L^\infty(\Omega)$ we get

$$\langle D_{p,\varphi}(u), u\rangle = \int_\Omega \varphi\left(x, y, |\nabla u|^{p-1}\right)|\nabla u|^p dx \geq m \int_\Omega |\nabla u|^p dx = m\|u\|^p_{W_0^{1,p}},$$

  which implies (K).
- If we assume that the condition (φ_{M}) holds then for any $u, v \in W_0^{1,p}(\Omega)$ and $y \in L^\infty(\Omega)$ we have

$$\begin{aligned}
\langle D_{p,\varphi}(u, y) - D_{p,\varphi}(v, y), u - v\rangle
&= \int_\Omega \left(\varphi\left(x, y, |\nabla u|^{p-1}\right)|\nabla u|^{p-2}\nabla u - \varphi\left(x, y, |\nabla v|^{p-1}\right)|\nabla v|^{p-2}\nabla v\right)\cdot(\nabla u - \nabla v)\,dx \\
&= \int_\Omega \varphi\left(x, y, |\nabla u|^{p-1}\right)|\nabla u|^{p-2}\cdot(|\nabla u|^2 - \nabla u\cdot\nabla v)dx \\
&\quad - \int_\Omega \varphi\left(x, y, |\nabla v|^{p-1}\right)|\nabla v|^{p-2}\cdot\left(\nabla u\cdot\nabla v - |\nabla v|^2\right)dx \\
&\geq \int_\Omega \varphi\left(x, y, |\nabla u|^{p-1}\right)|\nabla u|^{p-2}\cdot\left(|\nabla u|^2 - |\nabla u|\cdot|\nabla v|\right)dx \\
&\quad - \int_\Omega \varphi\left(x, y, |\nabla v|^{p-1}\right)|\nabla v|^{p-2}\cdot\left(|\nabla u|\cdot|\nabla v| - |\nabla v|^2)\right)dx \\
&= \int_\Omega \left(\varphi\left(x, y, |\nabla u|^{p-1}\right)|\nabla u|^{p-1} - \varphi\left(x, y, |\nabla v|^{p-1}\right)|\nabla v|^{p-1}\right)\cdot(|\nabla u| - |\nabla v|)\,dx > 0.
\end{aligned}$$

- Let us assume that the condition (φ_{S}) is satisfied. We assume that $p = 2$ and we define a function φ_0 by

$$\varphi_0(x, y, r) := \varphi(x, y, r) - \gamma$$

  for $x \in \Omega$, $y \in \mathbb{R}$ and $r \geq 0$. Next, we consider the operator D_{2,φ_0} with φ_0 instead of φ. Moreover, we put $\varphi_1(x, y, r) \equiv \gamma$. Then, the operator $D_{2,\varphi_1}(\cdot, y)$ associated with φ_1 is strongly monotone (it is the weak Laplace operator) uniformly for any $y \in L^\infty(\Omega)$. The above calculations provide that $D_{2,\varphi_0}(\cdot, y)$ is monotone for every fixed $y \in L^\infty(\Omega)$. Hence, the operator $D_{2,\varphi}(\cdot, y)$ satisfies $(\mathrm{M_s})$ with $m \equiv \gamma$.

  Now, we assume that $p > 2$. Let $u, v \in W_0^{1,p}(\Omega)$ and $y \in L^\infty(\Omega)$. The proof of the first part and Hölder's inequality yield

$$\begin{aligned}
\langle D_{p,\varphi}(u, y) - D_{p,\varphi}(v, y), u - v\rangle &\geq \int_\Omega \left(\varphi\left(x, y, |\nabla u|^{p-1}\right)|\nabla u|^{p-1} - \varphi\left(x, y, |\nabla v|^{p-1}\right)|\nabla v|^{p-1}\right)\cdot(|\nabla u| - |\nabla v|)\,dx \\
&\geq \int_\Omega \gamma\left(|\nabla u|^{p-1} - |\nabla v|^{p-1}\right)(|\nabla u| - |\nabla v|)\,dx \\
&= \gamma\left(\|u\|^p_{W_0^{1,p}} - \int_\Omega |\nabla u|^{p-1}\cdot|\nabla v|dx - \int_\Omega |\nabla v|^{p-1}\cdot|\nabla u|dx + \|v\|^p_{W_0^{1,p}}\right) \\
&\geq \gamma\left(\|u\|^p_{W_0^{1,p}} - \|u\|^{p-1}_{W_0^{1,p}}\|v\|_{W_0^{1,p}} - \|v\|^{p-1}_{W_0^{1,p}}\|u\|_{W_0^{1,p}} + \|v\|^p_{W_0^{1,p}}\right) \\
&= \gamma\left(\|u\|_{W_0^{1,p}} - \|v\|_{W_0^{1,p}}\right)\left(\|u\|^{p-1}_{W_0^{1,p}} - \|v\|^{p-1}_{W_0^{1,p}}\right).
\end{aligned}$$

□

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