Nonlocal heat equations with generalized fractional Laplacian

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Abstract

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1 Introduction

Recently, there has been much interest in attempts to better model diffusion processes taking into account their nonlinearity and, above all, nonlocality. The classical diffusion equation is a partial differential equation in which diffusion is expressed by a Laplacian. Solutions to this equation show an infinite speed of propagation of the disturbance, which clearly violates the laws of physics. In order to avoid this trouble, nonlinear PDEs of the form

$$u_t - \Delta u^{1+\delta} = \alpha u^p$$

[7] or some similar ones are applied. The nonlocality is taken into account by using the so-called fractional Laplacian. Mathematically the theory of this operator was very supportable, since there are a number of equivalent definitions of it (see [6]). However, diffusion equations with fractional Laplacian have solutions given in the whole space, so taking into account "anchoring" of the unknown function on the boundary of the given area was at least difficult. A certain solution to this problem is to use Spectral Theory, but a Laplacian defined in this way is no longer equivalent [9], in particular it

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has no stochastic interpretation. The approach of engineers to modelling phenomena is different. Here the main criterion for the value of the model is the correspondence between theoretical results and measurement values obtained from experiments, hence engineers often use equations in which it is possible to fit parameters to these results.

To this end, in paper [5] we applied Spectral Theory (see [2]) by applying an arbitrary function to the Laplace operator instead of a power function. Spectral Theory allows us to write any selfadjoint operator T (usually unbounded) on a Hilbert space by using a spectral integral

$$Tu = \int_{-\infty}^{+\infty} \lambda \, dE_\lambda u.$$

Then, for every measurable function g defined on the spectrum of this operator, a linear operator $g(T)$ is defined by a similar integral formula with $g(\lambda)$ instead of λ. When T is the Laplace operator Δ acting on $L^2(\Omega)$ and defined on the Sobolev space $H_0^1(\Omega)$ (a bounded open set $\Omega \subset \mathbb{R}^n$ with regular boundary), then the spectral integral trivializes to the sum of the series, because the spectrum here is a discrete sequence of positive numbers converging to infinity, which significantly simplifies considerations and does not constitute any restriction on the function g (all functions on the discrete set are measurable). In [5] we prove the existence of solutions of stationary equations with a generalized fractional Laplacian defined in this way and we study their regularity.

Here we develop the existence theory to the first order evolutionary equation with a diffusion operator given by this generalized fractional Laplacian and with a nonlinear term - a Nemytskii operator $u \mapsto f(t, x, u(t, x))$. We use two methods: in the next section, we replace the equation with a given initial condition by a fixed point problem in a certain Banach space of sequences of real valued continuous functions of time t. These functions are Fourier coefficients of u with respect to the complete orthonormal system of eigenfunctions of the Dirichlet Lapalacian. The existence of solutions is obtained due to the Schauder Fixed Point Theorem. Usually, the classical diffusion equation has the property that for any initial condition, a solution tends for the unique solution of the corresponding stationary problem. We prove the same for evolutionary equations with the generalized Dirichlet Laplacian, The next section is devoted to the semigroup method applied to our nonlocal diffusion equations. The assumptions are sligtly less restrictive but a solution obtained by this method – the mild solution – has a weaker sense than the ones given by the direct method.

The main novelty of this paper relies on the fact that the fractional Laplace operator given by a power function $z \mapsto z^\beta$ is replaced by a general function g defined on the spectrum of the Dirichlet Laplacian. We cannot compare our results with theorems obtained previously by many authors (comp. [10, 11, 12, 13]) since they used another definition of the ractional Laplacian. The spectral definition has been used in [4] however Idczak considered only linear equations.

2 Direct method for finding solutions

Consider the following initial-boundary value problem:

$$u_t + g(-\Delta)u = f(t,x,u), \qquad u(t,\partial\Omega) = 0, \qquad u(0,\cdot) = u_0, \tag{2.1}$$

where Ω is a bounded domain with Lipschitzian boundary, g is a positive function defined on the spectrum $(\lambda_n)_n$ of the Dirichlet Laplacian

$$-\Delta e_n(x) = \lambda_n e_n(x), \qquad e_n(\partial\Omega) = 0,$$

satisfying

$$\sum_n g(\lambda_n)^{-2} < \infty, \tag{2.2}$$

$f : [0,\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous w.r.t. t, measurable w.r.t. x and uniformly continuous w.r.t. u such that

$$|f(t,x,u)| \le a_0(t)a(x) + b|u|,$$

where $a_0 : [0,\infty) \to \mathbb{R}$ is continuous, $a \in L^2(\Omega)$, $b \ge 0$. The last inequality guarantees (the Krasnoselskii Theorem) that the Nemytskii operator

$$\mathbf{f} : L^2(\Omega) \ni u(t,\cdot) \mapsto f(t,\cdot,u(t,\cdot)) \in L^2(\Omega)$$

acts in this Hilbert space for any t and it is continuous. Assume that the initial function $u_0 \in L^2(\Omega)$. Since the family of normalized eigenfunctions $\{e_n :\ n \in \mathbb{N}\}$ consists an orthonormal basis in $L^2(\Omega)$, one can put

$$u(t,x) = \sum_n u_n(t)e_n(x)$$

in the equation and get

$$\sum_n \left(u_n'(t) + g(\lambda_n)u_n(t)\right) e_n(x) = \sum_n \left(\int_\Omega f(t,y,u(t,y))e_n(y)\,dy\right) e_n(x)$$

obviously under assumption all series are convergent in any sense and one can go with both operators on the left into the series. The equality implies the sequence

$$u_n'(t) + g(\lambda_n)u_n(t) = f_n(t)(u), \tag{2.3}$$

where

$$f_n(t)(u) := \int_\Omega f(t, y, u(t, y))e_n(y)\, dy.$$

From the initial condition we have

$$u_n(0) = \int_\Omega u_0(x)e_n(x)\, dx =: u_{n,0}.$$

One can inverse the left-hand side of (2.3) with this initial condition and get:

$$u_n(t) = u_{n,0}\mathrm{e}^{-g(\lambda_n)t} + \int_0^t \mathrm{e}^{-g(\lambda_n)(t-s)} f_n(s)(u)\, ds.$$

If the function f does not depend on u, then it gives the solution of the nonhomogeneous linear problem but, in general case, it is a splitting sequence of equations.

Fix $T > 0$ and search for solutions of (2.1) on the interval $[0, T]$. Let X be a Banach space of sequences of real continuous functions $u_n : [0, T] \to \mathbb{R}$, $n \in \mathbb{N}$, such that

$$\|u\|^2 := \sum_n \sup_{t\in[0,T]} |u_n(t)|^2 < \infty,$$

where $u = (u_n)_n$.

Theorem 1. (i) *X is a Banach space.*

(ii) *a set $\mathcal{A} \subset X$ is relatively compact if*

1° *for any $n \in \mathbb{N}$ there exists $M_n > 0$ such that $|u_n(t)| \le M_n$ for $t \in [0, T]$ and $u = (u_m)_m \in \mathcal{A}$;*

2° *for $u = (u_m)_m \in \mathcal{A}$ and any n, functions u_n are equicontinuous;*

3° *for any $\varepsilon > 0$, there exists n_0 such that*

$$\sum_{n=n_0+1}^{\infty} |u_n(t)|^2 \le \varepsilon$$

for $u = (u_m)_m \in \mathcal{A}$ and $t \in [0, T]$.

Proof. The proof of part (i) is standard. We shall prove part (ii). Take $\varepsilon > 0$. By assumption 3°, there exists n_0 such that

$$\sum_{n=n_0+1}^{\infty} |u_n(t)|^2 \le \frac{\varepsilon}{2}$$

for $u \in \mathcal{A}$ and $t \in [0,T]$. Due to assumptions 1°, 2° and the Ascoli-Arzéla Theorem the sets $\{u_n : u \in \mathcal{A}\}$ for $n = 1, 2, \dots, n_0$ are relatively compact in $C[0,T]$, hence they have finite $\frac{\varepsilon}{2n_0}$-nets $v_n^{(j)} \in C[0,T]$, $j = 1, \dots, p_n$, for any $u \in \mathcal{A}$ and $n \in \{1, \dots, n_0\}$, there exists j such that

$$\sup_t |u_n(t) - v_n^{(j_n)}(t)| \le \frac{\varepsilon}{2n_0}.$$

Now the sequences v having functions $v_n(j)$ for indices $j \le p_n$ and 0 in the remaining places constitute a finite set (there are $p_1 \cdot \ldots \cdot p_{n_0}$ sequences) such that $\|u - v\|^2 \le \varepsilon$ for any $u \in \mathcal{A}$. The sequence v from the finite set is chosen by $v_n = v_n^{(j_n)}$. $\square$

Let us notice that the space of continuous functions $[0,T] \to L^2(\Omega)$ seems to be more natural scene for our problem. A function from this space define

$$u(t,x) = \sum_n u_n(t) e_n(x)$$

and its norm

$$\sup_{t\in[0,T]} \|u(t,\cdot)\|_{L^2} = \sup_{t\in[0,T]} \|(u_n(t))\|_{l^2}$$

by the Parseval Identity. Space X is smaller so solutions obtained below will sit in $C([0,T], L^2(\Omega))$.

Consider an operator S defined on X by the formula:

$$(S(v))_n(t) := v_{0,n} \mathrm{e}^{-g(\lambda_n)t} + \int_0^t \mathrm{e}^{-g(\lambda_n)(t-s)} f_n(s)(v)\, ds$$

for $v = (v_m)_m \in X$. The fixed points u of S give solutions to (2.1)

$$u(t,x) = \sum_n u_n(t) e_n(x).$$

Theorem 2. (*i*) *Operator S takes values in X.*
(*ii*) *$S : X \to X$ is continuous.*
(*iii*) *Operator S is compact.*

Proof. The assumption (2.2) is crucial in the proof of (i). Notice that

$$|f_n(t)(u)| \le \|\mathbf{f}(u(t))\| \cdot \|e_n\| \le \sup_t a_0\|a\| + b\|u\| =: C(u).$$

But

$$\left|\int_0^t \exp(-g(\lambda_n)(t-s)) f_n(s)(u)\, ds\right| \le \int_0^t \exp(-g(\lambda_n)(t-s))|f_n(s)(u)|\, ds =: I_n(t)$$

and the right-hand side vanishes for $t = 0$, hence it takes the maximum at the point where its derivative equals 0 or at $t = T$. But the derivative

$$I_n'(t) = g(\lambda_n)\left(\frac{|f_n(t)(u)|}{g(\lambda_n)} - I_n(t)\right)$$

thus at this point

$$I_n(t) \le \frac{|f_n(t)(u)|}{g(\lambda_n)}.$$

If the maximum is reached at T, then $I_n'(T) \ge 0$ and

$$I_n(T) \le \frac{|f_n(T)(u)|}{g(\lambda_n)}.$$

Therefore

$$\sup_{t\in[0,T]}\left|\int_0^t \exp(-g(\lambda_n)(t-s)) f_n(s)(u)\, ds\right| \le \frac{\sup_{t\in[0,T]}|f_n(t)(u)|}{g(\lambda_n)}$$

and

$$\|S(u)\| \le \|u_0\| + C(u)\left(\sum_n \frac{1}{g(\lambda_n)^2}\right)^{1/2} < \infty$$

that proves (i).

Proof of (ii). Let $u^{(m)} \to u$ in X. Then $\|u^{(m)}\| \le R$. Take $\varepsilon > 0$ and let n_0 be such that

$$(2C(R))^2 \sum_{n=n_0+1}^{\infty} \frac{1}{g(\lambda_n)^2} < \frac{\varepsilon^2}{4}$$

so

$$\sum_{n=n_0+1}^{\infty} |S_n(u^{(m)})(t) - S_n(u)(t)|^2 < \frac{\varepsilon^2}{4}.$$

Now take $\delta > 0$ such that

$$|u^{(m)} - u| < \delta \Rightarrow |f(t,x,u^{(m)}) - f(t,x,u)| < \frac{\varepsilon}{\sqrt{2}n_0|\Omega|T}$$

for any t and x by the uniform continuity of f w.r.t. u. Take m_0 such that, for $m \geq m_0$, $t \in [0,T]$, $x \in \Omega$, $|u^{(m)}(t,x) - u(t,x)| < \delta$. Therefore, for $m \geq m_0$,

$$\|S(u^{(m)}) - S(u)\|^2 \leq \sum_{n=1}^{n_0} \sup_{t\in[0,T]} \left(\int_0^t \mathrm{e}^{-g(\lambda_n)(t-s)} |f_n(s)(u^{(m)}) - f_n(s)(u)|\, ds \right)^2 + \frac{\varepsilon^2}{4}$$

$$< T^2 n_0^2 \frac{\varepsilon^2}{2n_0^2 |\Omega|^2 T^2} \left(\int_\Omega e_n \right)^2 + \frac{\varepsilon^2}{4} = \varepsilon.$$

Proof of (iii). We should show that for any $u \in X$ with $\|u\| \leq R$ functions $S_n(u)$ are uniformly bounded and equicontinuous for every n and condition 3° in compactness criterion is satisfied. But the equiboundedness was proved before

$$|S_n(u)(t)| \leq |u_{n,0}| + T \sup_s |f_n(s)(u)| \leq |u_{n,0}| + TC(R)$$

and the equicontinuity follows from the boundedness of

$$S_n(u)'(t) = f_n(t)(u) - g(\lambda_n) \left(u_{n,0} \mathrm{e}^{-g(\lambda_n)t} + \int_0^t \mathrm{e}^{-g(\lambda_n)(t-s)} f_n(s)(u)\, ds \right).$$

Condition 3° is a consequence of the proof of (i). □

Theorem 3. *Under our assumptions if $b < 1$, then problem* (2.1) *has a solution.*

Proof. For any $u \in X$ we have

$$\|S(u)\| \leq \|u_0\| + \sup_{s\in[0,T]} a_0(t)\|a\| + b\|u\|$$

and the assumption $b < 1$ gives for sufficiently large R,

$$\|u_0\| + \sup_{s\in[0,T]} a_0(t)\|a\| + bR \leq R.$$

Therefore S maps the ball $B(0,R)$ into itself and the Schauder Fixed Point Theorem gives the existence of a solution to (2.1). □

Alternatively, one can use the Contraction Principle if f satisfies the Lipschitz condition

$$|f(t,x,u) - f(t,x,v)| \leq L|u - v|$$

for $t \in [0,T]$, $x \in \Omega$ and $u, v \in \mathbb{R}$ with $L < 1$ however we cannot omit the assumption (2.2) since without it values of S could leave beyond X. Thus the assumptions are more restrictive and the only advantage is the uniqueness of a solution.

The solutions obtained in the above theorems are global in time because $T > 0$ can be chosen arbitrarily. Solutions are of the C^1-class w.r.t. $t \in [0, \infty)$ and the regularity in the space variable ($u(t) \in H^p(\Omega)$) can be obtained only is some conditions about derivatives of $e_n \in C(\Omega)$ The last question is: when solutions to (2.1) tend to ground-state ones i.e. $\lim_{t\to\infty} u(t, x) = w(x)$, where w satisfies

$$g(-\Delta)w = f_\infty(x, w), \qquad w(\partial\Omega) = 0,$$

$f_\infty(x, w) := \lim_{t\to\infty} f(t, x, w)$ – the limit exists especially in the case f does not depend on t.

We have a partial answer for the third question:

Theorem 4. *Consider an evolutionary problem*

$$u_t + g(-\Delta)u = f(x, u), \qquad u(t, \partial\Omega) = 0, \quad u(0, \cdot) = u_0 \in L^2(\Omega), \tag{2.4}$$

end the corresponding stationary one

$$g(-\Delta)u = f(x, u), \qquad u(t, \partial\Omega) = 0. \tag{2.5}$$

Suppose that f satisfies the Lipschitz condition

$$|f(x, u) - f(x, v)| \le L|u - v|$$

for any $u, v \in \mathbb{R}$ and a.e. $x \in \Omega$, where

$$L < \beta := \inf\{g(\lambda_n) : n \in \mathbb{N}\}. \tag{2.6}$$

Then (2.5) *has the unique solution w and all solutions u to* (2.4) *tend to w in $L^2(\Omega)$ as $t \to +\infty$:*

$$\lim_{t\to+\infty} \|u(t, \cdot) - w\| = 0.$$

Proof. We shall estimate the derivative

$$\mathcal{L} := \frac{d}{dt}\|u(t) - w\|^2,$$

where the square of the L^2-norm of the function equals the sum of squares of Fourier's coefficients. Thus

$$\mathcal{L} = 2\sum_n (u_n(t) - w_n) \cdot u_n'(t).$$

But

$$u_n'(t) = -g(\lambda_n)\left(u_{n,0} + \int_0^t \mathrm{e}^{g(\lambda_n)s} f_n(s)(u)\, ds\right)\mathrm{e}^{-g(\lambda_n)t} + f_n(t)(u)$$

$$= -g(\lambda_n)u_n(t) + f_n(t)(u) = -g(\lambda_n)(u_n(t) - w_n) + (f_n(t)(u) - f_n(w)),$$

since $w_n = f_n(w)/g(\lambda_n)$. Hence

$$\mathcal{L} = -2\sum_n g(\lambda_n)(u_n(t) - w_n)^2 + 2\sum_n (u_n(t) - w_n)\cdot(f_n(t)(u) - f_n(w))$$

$$\leq -2\beta\|u(t) - w\|^2 + 2L\|u(t) - w\|^2.$$

Therefore

$$\frac{d}{dt}\ln\|u(t) - w\|^2 \leq -2(\beta - L)$$

and

$$\|u(t) - w\|^2 \leq \|u_0 - w\|^2 \exp(-2(\beta - L)t)$$

that gives the assertion.

In fact, we prove that $u(t)$ tends to w as $t \to +\infty$ exponentially. □

3 Applications of Compact Semigroups

Let g be a real function defined on the spectrum $(\lambda_n)_{n\in\mathbb{N}}$ of the Dirichlet Laplacian such that $\lim_{n\to\infty} g(\lambda_n) = +\infty$. For any $t \geq 0$ we define a family of linear operator

$$T(t)u = \sum_{n=1}^{\infty} \mathrm{e}^{-g(\lambda_n)t}\langle u, e_n\rangle\, e_n \qquad \text{for } u \in L^2(\Omega). \tag{3.1}$$

It easy to see that $T(t)$ is linear and since $g(\lambda_n) \to +\infty$ there is a number $\omega \in \mathbb{R}$ such that $\|T(t)\| \leq \mathrm{e}^{\omega t}$ for $t \geq 0$. Moreover, for any $x \in X$ we have

$$\|T(t)u - u\|^2 = \sum_{n=1}^{\infty} |\mathrm{e}^{-g(\lambda_n)t} - 1|^2 |\langle u, e_n\rangle|^2 \leq \mathrm{e}^{2\omega t}\|u\|^2,$$

hence

$$\lim_{t\to 0^+}\|T(t)u - u\| = 0,$$

which means that $\{T(t)\}_{t\geq 0}$ is C_0-semigroups on $L^2(\Omega)$.

Now, we will show that $A = -g(-\Delta)$ is infinitesimal generator of the semigroup $\{T(t)\}_{t\geq 0}$. Recall that the operator $A\colon \operatorname{Dom}(A) \to X$ with domain

$$\operatorname{Dom}(A) = \left\{x \in X : \lim_{t\to 0^+} \frac{T(t) - x}{t} \text{ exists}\right\}$$

given by

$$Ax := \lim_{t\to 0^+} \frac{T(t) - x}{t}$$

is called a *infinitesimal generator of the semigroup* $\{T(t)\}_{t\geq 0}$ in a Banach space X.

Proposition 1. *Let A be the infinitesimal generator C_0–semigroup $\{T(t)\}_{t\geq 0}$, which defined by (3.1). Then $A = -g(-\Delta)$.*

Proof. We first prove that

$$\lim_{t\to 0^+} \left\| \frac{T(t)u - u}{t} + g(-\Delta) \right\| = 0$$

for all $u \in \operatorname{Dom} g(-\Delta)$.

It is easily than

$$\begin{aligned}\left\| \frac{T(t)u - u}{t} + g(-\Delta)u \right\|^2 &= \left\| \sum_{n=1}^{\infty} \left(\frac{\mathrm{e}^{-tg(\lambda_n)} - 1}{t} + g(\lambda_n) \right) \langle u, e_n \rangle \, e_n \right\|^2 \\ &= \sum_{n=1}^{\infty} \left| \frac{\mathrm{e}^{-tg(\lambda_n)} - 1}{t} + g(\lambda_n) \right|^2 |\, \langle u, e_n \rangle \,|^2.\end{aligned}$$

Since $g(\lambda_n) \to \infty$ there is a number $m \in \mathbb{N}$ such that $g(\lambda_n) > 0$ for all $n \geq m$. Then we see at once that

$$\left| \frac{\mathrm{e}^{-tg(\lambda_n)} - 1}{t} + g(\lambda_n) \right| = g(\lambda_n) \cdot \left| \frac{\mathrm{e}^{-tg(\lambda_n)} - 1}{tg(\lambda_n)} + 1) \right| < g(\lambda_n)$$

for $n \geq m$, which is clear from boundedness of function $(0, \infty) \ni x \mapsto \frac{\exp(-ax) - 1}{ax} + 1$ with parameters $a > 0$. Hence

$$\sum_{n=1}^{\infty} \left| \frac{\mathrm{e}^{-tg(\lambda_n)} - 1}{t} + g(\lambda_n) \right|^2 |\, \langle u, e_n \rangle \,|^2$$

is uniformly convergent, because $u \in \operatorname{Dom} g(-\Delta)$. Therefore we can pass to the limit inside above series

$$\lim_{t\to 0^+} \sum_{n=1}^{\infty} \left| \frac{\mathrm{e}^{-tg(\lambda_n)} - 1}{t} + g(\lambda_n) \right|^2 |\, \langle u, e_n \rangle \,|^2 = 0$$

and in consequence $A|_{\mathrm{Dom}\, g(-\Delta)} = g(-\Delta)$. Since $T(t)$ is self-adjoint for any $t \geq 0$, then so is the generator A. It is known that self-adjoint does not have self-adjoint extensions, which gives $A = -g(-\Delta)$. $\square$

A semigroup $\{T(t)\}_{t\geq 0}$ is said to be *compact* if $T(t)$ is compact for each $t > 0$.

It is easy to check that the semigroup (3.1) is compact, provided by $g(\lambda_n) \to \infty$.

We begin in abstract setting. We consider the following semilinear initial value problem

$$\mathbf{u}'(t) = A\mathbf{u}(t) + \mathbf{f}(t, \mathbf{u}(t)), \qquad \mathbf{u}(0) = \mathbf{u}_0, \tag{3.2}$$

where A is the infinitesimal generator of C_0–semigroup $\{T(t)\}_{t\geq 0}$ and $\mathbf{f}\colon X \times [0, +\infty) \times X \to X$ is continuous.

A solution $\mathbf{u} \in C([0, +\infty), X)$ of the integral equation

$$\mathbf{u}(t) = T(t)\mathbf{u}_0 + \int_0^t T(t-s)\mathbf{f}(s, \mathbf{u}(s))\, ds \tag{3.3}$$

is called a *mild solution* of the initial value problem (3.2).

For semilinear equations with compact semigroup we have the following local existence theorem.

Theorem 5. *([8]) Let A be the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$. If $\mathbf{f}\colon [0, +\infty) \times U \to X$ is continuous, where $U \subset X$ is open then for every $\mathbf{u}_0 \in U$ there exists a $t_1 \in (0, +\infty)$ such that the initial value problem (3.2) has a mild solution $\mathbf{u} \in C([0, t_1,], X)$.*

To obtain the existence of global mild solutions of the problem (3.2) we will use the following conditions.

Theorem 6. *([8]) Let A be the infinitesimal generator of a compact semigroup $\{T(t)\}_{t\geq 0}$. Let $\mathbf{f}\colon [0, +\infty) \times X \to X$ be continuous and maps bounded sets in $[0, +\infty) \times X$ into bounded sets in X. Then for every $u_0 \in X$ the initial value problem (3.2) has a global solution $\mathbf{u} \in C([0, +\infty), X)$ if there exist two locally integrable functions $k_1, k_2\colon [0, +\infty) \to [0, +\infty)$ such that*

$$\|\mathbf{f}(t, \mathbf{u})\| \leq k_1(t)\, \|\mathbf{u}\| + k_2(t) \qquad \text{for } t \in [0, +\infty), \mathbf{u} \in X. \tag{3.4}$$

Now, we consider again the problem (2.1). We assume that $f\colon [0, +\infty) \times \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous w.r.t. $(t, u) \in [0, +\infty) \times \mathbb{R}$ for a.e. $x \in \Omega$, measurable w.r.t. $x \in \Omega$ for each $t \in [0, +\infty)$ and $u \in \mathbb{R}$ and

$$|f(t, x, u)| \leq a_0(t)a(x) + b|u|, \tag{3.5}$$

where $a_0\colon [0, +\infty) \to [0, +\infty)$ is continuous, $a \in L^2(\Omega)$ and $b \geq 0$.

Theorem 7. *Under the above assumptions, the problem (2.1) has a global mild solution.*

Proof. Putting $A = -g(-\Delta)$ and defining nonlinear mapping $\mathbf{f}(t, u) = f(t, \cdot, u(\cdot))$ for $t \in [0, +\infty)$, $u \in L^2(\Omega)$, the initial-boundary value problem (2.1) can be rewritten as

$$\mathbf{u}'(t) = A\mathbf{u}(t) + \mathbf{f}(t, \mathbf{u}(t)), \qquad \mathbf{u}(0) = u_0. \tag{3.6}$$

It is sufficient to show that $\mathbf{f}\colon [0, +\infty) \times L^2(\Omega) \to L^2(\Omega)$ is well defined and continuous. By (3.5) $\mathbf{f}$ is well defined, therefore we have to show only continuity of $\mathbf{f}$.

Let $u_n \to u$ in $L^2(\Omega)$ and $t_n \to t$ in $[0, +\infty)$ and we will prove that $f(t_n, \cdot, u_n(\cdot)) \to f(t, \cdot, u(\cdot))$ in $L^2(\Omega)$. On the contrary suppose that there exists a number $\varepsilon > 0$ such that

$$\int_\Omega |f(t_n, x, u_n(x)) - f(t, x, u(x))|^2 \, dx > \varepsilon. \tag{3.7}$$

On the other hand, we can extract a subsequence $(u_{n_k})_{k\in\mathbb{N}}$ converged to u a.e., also there is a function $h \in L^2(\Omega)$ such that $|u_{n_k}(x)| \le h(x)$ for each $k \in \mathbb{N}$ and a.e. $x \in \Omega$. Then

$$\begin{aligned} |f(t_{n_k}, x, u_{n_k}(x))| &\le a_0(t_{n_k})a(x) + b|u_{n_k}(x)| \\ &\le \sup_{k\in\mathbb{N}} a_0(t_{n_k}) \cdot a(x) + bh(x) =: \psi(x) \in L^2(\Omega). \end{aligned}$$

Hence and Jensen's inequality we get

$$\begin{aligned} |f(t_{n_k}, x, u_{n_k}(x)) - f(t, x, u(x))|^2 &\le 2\Big(|f(t_{n_k}, x, u_{n_k}(x))|^2 + |f(t, x, u(x))|^2\Big) \\ &\le 2\Big(\psi(x)^2 + |f(t, x, u(x))|^2\Big) \end{aligned}$$

for every $k \in \mathbb{N}$ and a.e. $x \in \Omega$. Continuity of f w.r.t (t, u) yields $f(t_{n_k}, x, u_{n_k}(x)) \to f(t, x, u(x))$ a.e. Using Lebesgue's Dominated Convergence Theorem, we obtain

$$\int_\Omega |f(t_{n_k}, x, u_{n_k}(x)) - f(t, x, u(x))|^2 \, dx \to 0,$$

which contradicts (3.7) and in consequence, $\mathbf{f}$ is continuous.

By application of the Theorem 5 we proved existence a local mild solution. Furthermore, the assumptions of Theorem 6 are satisfied, as is easy to check. Hence the mild solution is global. □

Now, we give a sufficient condition for a mild solution to be a classical solution.

Theorem 8. *([8]) Let A be the infinitesimal generator of a C_0–semigroup $\{T(t)\}_{t\leq 0}$ on X. If $\mathbf{f}\colon [0,+\infty)\times X\to X$ is continuously differentiable from $[0,+\infty)\times X$ into X then the mild solution of (3.2) with $\mathbf{u}_0\in \operatorname{Dom}(A)$ is a classical solution of the initial value problem.*

We next make some additional conditions for the function f. Let us assume that $f\colon [0,+\infty)\times\Omega\times\mathbb{R}\to\mathbb{R}$ is C^1 w.r.t. $(t,u)\in[0,+\infty)\times\mathbb{R}$ for a.e. $x\in\Omega$ and

$$|f_t(t,x,u)|\leq a_1(t)c(x)+d|u|, \tag{3.8}$$

$$|f_u(t,x,u)|\leq a_2(t), \tag{3.9}$$

for $t\geq 0$, a.e $x\in\Omega$, $u\in\mathbb{R}$, where $a_1,a_2\colon [0,+\infty)\to[0,+\infty)$ are continuous, $c\in L^2(\Omega)$ and $d\geq 0$.

Theorem 9. *Assume that the above conditions for the function f hold. If $u_0\in \operatorname{Dom} g(-\Delta)$ then the problem (2.2) has a solution, which is C^1-class w.r.t. $t\in[0,+\infty)$.*

Proof. It suffices to prove that $\mathbf{f}\colon [0,+\infty)\times L^2(\Omega)\to L^2(\Omega)$ is C^1.

Now we only need to show that $\mathbf{f}_t$ and $\mathbf{f}_u$ are continuous. Let us fix $t\geq 0$ and $u\in L^2(\Omega)$. We define a linear mapping $T\colon\mathbb{R}\to L^2(\Omega)$ as follows

$$Ts := f_t(t,\cdot,u(\cdot))s.$$

for any $s\in\mathbb{R}$. Then for arbitrary $s\in\mathbb{R}$ by (3.8) we obtain

$$\left(\int_\Omega |(Ts)(x)|^2\,dx\right)^{1/2}\leq\left(\int_\Omega |f_t(t,x,u(x)s|^2\,dx\right)^{1/2}\leq\Big(a_1(t)\|c\|+d\|u\|\Big)|s|,$$

hence T is well-defined and bounded operator between $\mathbb{R}$ and $L^2(\Omega)$. Since

$$\begin{aligned} f(t+s,x,u(x))-f(t,x,u(x)) &= \int_0^1 \frac{d}{d\tau}\Big(f(t+\tau s,x,u(x))\Big)\,d\tau\\ &= \int_0^1 f_t(t+\tau s,x,u(x))s\,d\tau \end{aligned}$$

we get

$$\|\mathbf{f}(t+s,u)-\mathbf{f}(t,u)-Ts\|^2=\int_\Omega\left|\int_0^1\Big(f_t(t+\tau s,x,u(x))-f_t(t,x,u(x))\Big)s\,d\tau\right|^2dx.$$

Using Jensen's inequality and Fubini theorem yield

$$\|\mathbf{f}(t+s,u)-\mathbf{f}(t,u)-Ts\|^2 \leq \int_0^1 \left(\int_\Omega |f_t(t+\tau s,x,u(x))-f_t(t,x,u(x))|^2\,dx \right) d\tau \cdot s^2.$$

Therefore

$$\frac{\|\mathbf{f}(t+s,u)-\mathbf{f}(t,u)-Ts\|}{|s|} \leq \left(\int_0^1 \|f_t(t+\tau s,\cdot,u(\cdot))-f_t(t,\cdot,u(\cdot))\|^2\,d\tau \right)^{1/2}$$

and $\mathbf{f}_t(t,u)=T$.

Now, we shall show that $(t,u)\mapsto \mathbf{f}_t(t,u)$ is continuous. Since

$$\left\| \Big(\mathbf{f}_t(t,u)-\mathbf{f}_t(t_0,u_0) \Big) s \right\| = |s| \|f_t'(t,\cdot,u(\cdot))-f_t'(t_0,\cdot,u_0(\cdot))\|$$

we have

$$\|\mathbf{f}_t(t,u)-\mathbf{f}_t(t_0,u_0)\| \leq \|f_t'(t,\cdot,u(\cdot))-f_t'(t_0,\cdot,u_0(\cdot))\|$$

hence $\mathbf{f}_t$ is continuous.

The derivative of $\mathbf{f}$ at point (t,u) with respect to u (compare [3]) is defined by

$$\mathbf{f}_u(t,u)(v) := f_u(t,\cdot,u(\cdot))v(\cdot)$$

for $v\in L^2(\Omega)$ and mapping $(t,u)\mapsto \mathbf{f}_t(t,u)$ is continuous. □

4 Numerical simulations

In the last section, we present some numerical simulations of solutions to our evolutionary problem suggesting how they depend on function g responsible for the generalized fractional Laplacian and on the other variables: function f and the initial value u_0. In particular, we are interested in the asymptotic behavior of solutions when the conditions of Theorem 4 are not satisfied, for example, when there is more than one a stationary solution.

We will use the simplest method for finding an approximate solution – a partial sum of the Fourier series

$$u(t,x)=\sum_n u_n(t)e_n(x).$$

One can find the explicit formulas for all functions if the right-hand side f has the simple form $bu+f(x)$ and $\Omega := (0,\pi)\subset\mathbb{R}$. We have

$$e_n(x) := \sqrt{\frac{2}{\pi}}\sin nx,$$

$$u_n(t) := \left(u_{n,0} - \frac{f_n}{g(n^2) - b}\right) \exp(-(g(n^2) - b)t) + \frac{f_n}{g(n^2) - b},$$

where notations are from section 2 and f_n is the n-th Fourier coefficient of $x \mapsto f(x)$. All pictures below are obtained by using Python.

First we take $f(x) \equiv 1$, $b = 0$, $u_0(x) = x(\pi - x)$ and we change function $g := z$ (the usual Laplacian – the second derivative), $= z^{0.6}$ (the fractional Laplacian) and $= \sin^2 z$.

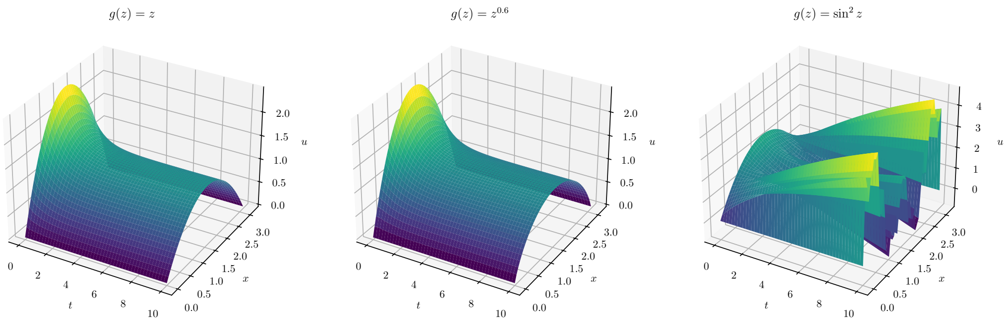

Figure 1: Solutions to (2.1) for $f(x) \equiv 1$, $b = 0$, $u_0(x) = x(\pi - x)$.

One can see that plots (Figure 1) for fractional and usual Laplacians are very similar but the last one is different. It is not surprising since $g(n^2)$ is a bounded sequence with the infimum of values exactly 0. If we change the initial function u_0, we get very similar situation. Obviously, if we replace f, then stationary solutions for usual and fractional Laplacian will change but the asymptotic behavior will be the same. If we change $b = 2$ then the assumptions of Theorem 4 will fail – the Lipschitz constant $L = b$ is now greater than the infimum of g on the spectrum. It is not surprising that solutions (Figure 2) blow up for large t : the first summand tends to the infinity exponentially and it cannot compensated by the remaining series which is bounded.

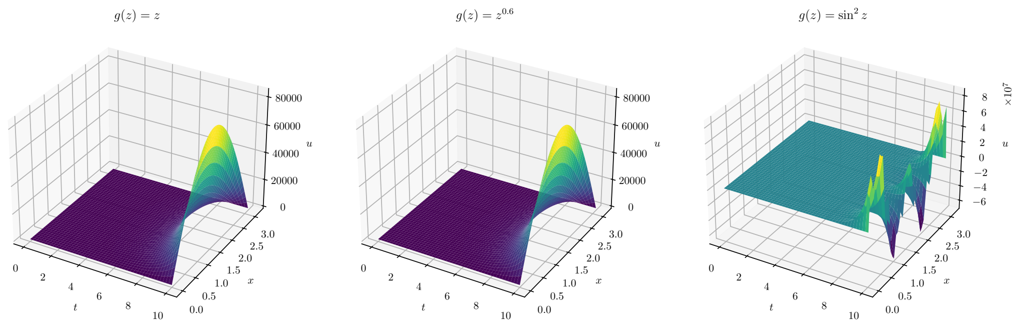

Figure 2: Solutions to (2.1) for $f(x) \equiv 1$, $b = 2$, $u_0(x) = x(\pi - x)$.

The choice $b = 4$ gives the stationary equation that has no solutions vanishing on $\partial\Omega$ for all f. This solutions exists only for f orthogonal to e_2, $f(x) = 1 - \cos 2x$, for instance. If the remaining parameters are as above the pictures look as follows:

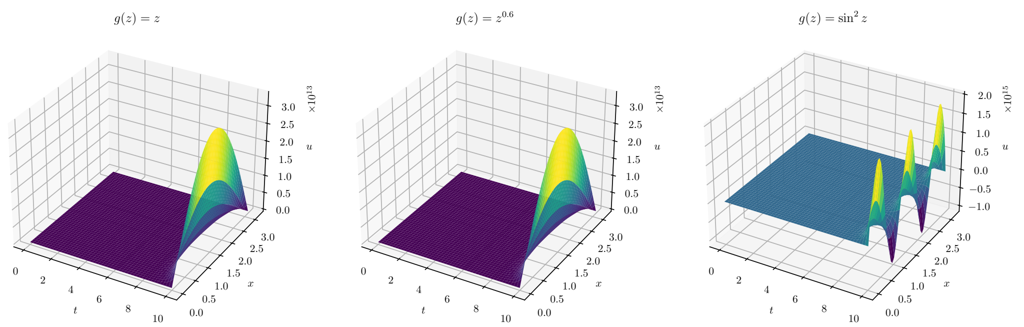

Figure 3: Solutions to (2.1) for $f(x) = 1 - \cos 2x$, $b = 4$, $u_0(x) = x(\pi - x)$.

It seems that solutions (Figure 3) tend to $\pm\infty$ at almost all x. Similar pictures appear for other choice of u_0 and $f \perp e_2$.

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