Water Resources Research

Supporting Information for

Unifying Advective and Diffusive Descriptions of Bedform Pumping in the Benthic Biolayer of Streams

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Introduction. This Supplemental Information includes mathematical derivations (Text S1 through S3) related to the bedform pumping model presented in the main text.

Text S1: Derivation of the Convolution Representation of Flux Across the SWI. In this section we derive from the BPM the convolution representation of advective flux across the SWI (equation (1a) in the main text). A striking feature of the BPM's two-dimensional velocity field is that, along any streamline, the x-component of the velocity is constant and equal to $u_x(\bar{x}_0)$, where $\bar{x} = \bar{x}_0$ is the dimensionless location along the SWI where the streamline first enters the streambed (see proof in the SI of Grant et al., 2014); because streamlines in a unit cell are symmetric, the same streamline exits the sediment bed at $\bar{x} = -\bar{x}_0$ (see expanded view in **Figure 1c** in the main text). We can utilize these two features of the BPM's flow field to solve for the unsteady mass flux across the SWI, in the case where the water column concentration is a function of time. Letting W [L] represent the width of the stream, the rate at which mass flows into the streambed across a differential area, $W dx_0$, is: $d\dot{m}_{\text{in}}(x_0,t) = C_w(t) u_y(x = x_0, y = 0) dx_0 W$ where $C_w(t)$ is the concentration of the solute in the overlying water column at time t (assumed not to vary over the length of a single unit cell) and $u_y(x = x_0, y = 0)$ is the vertical velocity of water parcels crossing into the sediment from the stream. Likewise, if $C_f(t;x_0)$ represents the final solute concentration at time t on the streamline that entered the streambed at $x = x_0$, the rate at which mass flows out of the streambed is: $d\dot{m}_{\text{out}}(x_0,t) = C_f(t;x_0) u_y(x = -x_0, y = 0) dx_0 W$. Taking the difference of these two mass flow rates, substituting equation (R2) in **Figure 1** (main text) for the y-velocity at the SWI, integrating over all streamlines in the unit cell, and dividing by the unit cell's interfacial area, we arrive at equation (S1) for the average flux of solute across the SWI at any time t.

$$J(t) = \frac{u_m}{\pi}\left[C_w(t) - \int_0^{\pi/2} C_f(t;\bar{x}_0)\sin\bar{x}_0 \, d\bar{x}_0\right] \tag{S1}$$

As written, equation (S1) is not particularly useful, because the integral on the right-hand side is expressed in terms of an unknown final concentration, $C_f(t;\bar{x}_0)$. However, if the solute is conservative, the final concentration at time t must equal the concentration in the overlying water column at time, $t - \tau(\bar{x}_0)$, where $\tau(\bar{x}_0)$ is the streamline-dependent residence time; i.e., the time a solute spends traveling along a streamline from its starting position, $\bar{x} = \bar{x}_0$, to its ending position, $\bar{x} = -\bar{x}_0$: $C_f(t;\bar{x}_0) = C_w(t - \tau(\bar{x}_0))$.

The constant nature of the BPM's x-velocity along any streamline implies that the streamline's residence time can be estimated from the x-distance a water parcel travels along the streamline divided by the fixed x-component of the velocity associated with that streamline where θ is streambed porosity:

$$\tau = \frac{-2x_0}{u_x(x_0)/\theta} = \frac{\lambda \bar{x}_0}{\pi u_m \cos \bar{x}_0/\theta} \tag{S2}$$

From these results, the flux across the SWI can be expressed solely as a function of the overlying water column concentration:

$$J(t) = \frac{u_m}{\pi}\left[C_w(t) - \int_0^{\pi/2} C_w\left(t - \frac{\lambda \bar{x}_0 \theta}{\pi u_m \cos \bar{x}_0}\right) \sin \bar{x}_0 \, d\bar{x}_0\right] \tag{S3}$$

Following the addition of solute to the water column, streamlines in the unit cell shown in **Figure 1c** (main text) can be divided into two groups: (1) those for which solute has already transported the full length of the streamline (i.e. the solute has "broken through" the streamline and is returning to the stream); and (2) those for which solute has not yet broken through. At the boundary is a critical streamline, denoted by its starting x-position at the SWI ($\bar{x} = \bar{x}_{0,c}$), that separates the former ($0 \le \bar{x} \le \bar{x}_{0,c}$) from the latter ($\bar{x}_{0,c} < \bar{x} \le \pi/2$) (Elliott and Brooks, 1997a). Because the solute concentration is zero at the terminus of stream lines in the second group (i.e., $C_f(t;\bar{x}_0) = 0$ for $\bar{x}_{0c} < \bar{x}_0 < \pi/2$), the upper limit of the integral in equation (S3) can be adjusted downward:

$$J(t) = \frac{u_m}{\pi}\left[C_w(t) - \int_0^{\bar{x}_{0c}} C_w\left(t - \frac{\lambda \bar{x}_0 \theta}{\pi u_m \cos \bar{x}_0}\right) \sin \bar{x}_0 \, d\bar{x}_0\right] \tag{S4}$$

Performing a change integration variable from $\bar{x}_0$ to $\bar{\tau}$ (utilizing the relationship between these two variables, see equation (S2)) we obtain the convolution representation of the BPM's residence time distribution presented in the main text (equation (1a)).

Text S2: Coherence of Our and EB's Definition of the BPM's RTD. Our definition of the RTD's CDF (equation (2b)) is superficially different from the one derived for the BPM by Elliott and Brooks (hereafter, EB) (Elliott and Brooks, 1997a). Here, we adopted the standard definition for the CDF of an RTD, $F_{\mathrm{RTD}}(\bar{\tau})$, as the fraction of solute entering the sediment bed in a short time near $t = 0$ and exiting the bed by time τ (Fogler, 2016). EB, on the other hand, defined their RTD function, $\bar{R}(\tau)$, as "the fraction of solute which entered the bed in a short time near $t = 0$ and remains in the bed at time τ" (Elliott and Brooks, 1997a). For a conservative solute that enters the sediment near $t = 0$, by time $t = \tau$ the solute is either still in the bed or has exited the bed; i.e., there is no other place it could be. Thus, our two RTD definitions must sum to unity: $\bar{R}(\bar{\tau})+F(\bar{\tau})=1$. Subsituting equation (2b) and rearranging, we arrive at EB's solution for their RTD, $\bar{\tau}=\cos^{-1}\left[\bar{R}(\bar{\tau})\right]/\bar{R}(\bar{\tau})$ (see equation (21c) in Elliott and Brooks (1997a)), where our dimensionless time $\bar{\tau}$ is equivalent to EB's $t^*/2\theta$ and $t^*=\left(2\pi u_m/\lambda\right)t$. Hence, our RTD is mathematically coherent with EB's RTD.

Text S3. Derivation of the BPM's Residence Time Function. In this section we derive equation (8a) in the main text, which represents the time $\tau(\bar{x},\bar{y})$ a water parcel requires to travel from the point where it enters the bed at the SWI to any location $(\bar{x},\bar{y})$ in the sediment. We begin by defining a stream function $\psi(x,y)$ for the BPM (Sabersky and Acosta, 1989):

$$u_x=-\frac{\partial\psi}{\partial y} \tag{S5a}$$

$$u_y=\frac{\partial\psi}{\partial x} \tag{S5b}$$

In these equations u_x and u_y represent the BPM's Darcy fluxes in the x- and y-directions, respectively. Substituting the BPM velocity components $u_x=-u_m\cos\bar{x}e^{-y}$ and $u_y=u_m\sin\bar{x}e^{-y}$ (see **Figure 1** in the main text, where u_m is the maximum Darcy flux across the SWI) and integrating the resulting differential equations we arrive at the following stream function for the BPM:

$$\psi(\bar{x},\bar{y})=-\frac{\lambda u_m}{2\pi}\cos\bar{x}e^{-\bar{y}} \tag{S6}$$

Streamlines are obtained by setting the stream function equal to a constant, $\psi(\bar{x},\bar{y})=C_1$. The difference between any two stream function constants $\Delta\psi=C_2-C_1$ represents the volumetric flow rate per unit width of sediment bed [$m^3m^{-1}s^{-1}$] flowing between the streamlines represented by $\psi(\bar{x},\bar{y})=C_1$ and $\psi(\bar{x},\bar{y})=C_2$. In the case of the BPM's flow field, a stream function's constant can be written in terms of the dimensionless horizontal position ($\bar{x}=\bar{x}_0$) where the streamline in question first crosses the sediment-water interface (at $\bar{y}=0$) in the downwelling zone:

$$C=-\frac{\lambda u_m}{2\pi}\cos\bar{x}_0,\ 0<\bar{x}_0<\pi/2 \tag{S7}$$

Combining equations (S6) and (S7), we arrive at the following implicit equation for the streamline that intersects the sediment-water interface in the downwelling zone at $\bar{x}=\bar{x}_0$:

$$\cos\bar{x}_0=\cos\bar{x}e^{-\bar{y}},\ 0<\bar{x}_0<\pi/2 \tag{S8}$$

For the unit cell $-\pi/2 \le \bar{x}_0 \le \pi/2$ (see **Figure 1c** in the main text), each streamline begins and ends at $\bar{x}_0$ and $-\bar{x}_0$, respectively. Thus, the age of a water parcel at any position $\bar{x}$ can be calculated from the ratio of the $x-$distance traveled, $(\bar{x}-\bar{x}_0)\lambda/2\pi$, and the constant x-component of the water parcel's velocity $u_x(\bar{x}_0,\bar{y}=0)/\theta$ (see discussion of the BPM's flow field in **Text S1**) where θ denotes sediment porosity:

$$\tau(\bar{x}|\bar{x}_0)=\frac{(\bar{x}-\bar{x}_0)\lambda/2\pi}{u_x(\bar{x}_0,\bar{y}=0)/\theta},\ 0<\bar{x}_0<\pi/2 \quad \text{(S9)}$$

The notation $\tau(\bar{x}|\bar{x}_0)$ denotes the age of a water parcel located at position $\bar{x}$ along the streamline that enters the streambed at position $\bar{x}_0$. We would like to eliminate the starting position of the streamline, $\bar{x}_0$, from equation (S9). To that end, an expression for $\bar{x}_0$ can be obtained by rearranging the equation for a streamline (equation (S8)):

$$\bar{x}_0=\cos^{-1}\left(\cos\bar{x}e^{-\bar{y}}\right) \quad \text{(S10)}$$

Substituting equations (S10) and (R3) (**Figure 1**) into equation (S9) we obtain equation (8a) in the main text:

$$\bar{\tau}(\bar{x},\bar{y})=\frac{\tau(\bar{x},\bar{y})}{t_T}=\frac{\cos^{-1}\left(\cos\bar{x}e^{-\bar{y}}\right)-\bar{x}}{2\cos\bar{x}e^{-\bar{y}}},\ t_T=\frac{\lambda\theta}{\pi u_m}$$

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