An operative approach to solve Homogeneous differential–anti-differential equations

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Abstract

In this work, we extend the theory of differential equations through a new way. To do this, we give an idea of differential–anti-differential equations and define ordinary as well as partial derivative–anti-derivative operator with a base function to solve several types of such equations. The operator is applied to construct several Auxiliary equations for a Homogeneous differential–anti-differential equations. The roots, of the Auxiliary equations, are then inserted in the base function to get exact solutions of the corresponding equations. The process can be used to solve both Homogeneous linear and non-linear ordinary as well as partial differential–anti-differential equations. The technique has special property that it can solve several different types of differential equations including continuity, Heat, Wave, Laplace, Schrodinger, Euler, Blasius differential equations.

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1 Introduction

Differential equations is a branch of mathematics introduced in the mid of 17th century by Leibniz and Newton when they were studying geometry and mechanics. It not only connects the branches of mathematics but also connects other branches

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of science with mathematics. Linear as well as non-linear homogeneous differential equations play an important role in different fields of science and especially in Physics, Quantum Mechanics, Fluid Mechanics, and General, Special Relativity etc. There are several types of differential equations with no special method to solve all of them. Some equations can be solved analytically while the other can be solved numerically. For some solving techniques, we refer readers to Faith (2018); Nasr *et al.* (2020); Al. Din (2020), Islam *et al.* (2010); Bronstein & Manuel (1094); Boyce *et al.* (2021); Ince (1956); Murphy (2011) etc. Among the methods given in the above works, the Auxiliary or Characteristics equation method is very simple and easy to find the exact solution of linear homogeneous ordinary differential. We extend this method to solve some linear and non-linear ordinary and partial differential–anti-differential equations. The extension of this method is to be called Characteristic Method for linear and non-linear ordinary and partial differential–anti-differential equations. This method helps to generate Auxiliary or Characteristics equations for Linear and non-linear homogeneous ordinary and partial differential–anti-differential equations and even for fractional differential equations also. Moreover, by using this method, one can generate more than one Auxiliary or Characteristic equations which help to find the exact solutions easily.

The objective of our work is to extend the theory of differential equations. In this regards, we define Homogeneous ordinary and partial differential–anti-differential equations and give a simple and short method to solve several types of these equations. To do this, we introduce an ordinary as well as partial derivative–anti-derivative operator $D_{x^n} = \frac{\partial^n}{\partial x^n}$, $n \in \mathbb{Z}$, with a base function having some parameters. Use of this operator in the Homogeneous differential–anti-differential equations leads to their Auxiliary equation. Roots of the Auxiliary equation are values of the parameters in the base function. By inserting these values in the base function, we get exact solutions of the corresponding differential–anti-differential equation. Note that, if n is positive integer then $\frac{\partial^n}{\partial x^n}$ is a partial derivative with respect to the independent variable x while, if n is negative then $\frac{\partial^n}{\partial x^n}$ is a partial anti-derivative with respect to x. Moreover, if the base function depends on several independent variables then $\frac{\partial^n}{\partial x^n}$ is a partial derivative/anti-derivative, while if the base function is a function of one variable, then $\frac{\partial^n}{\partial x^n}$ is in fact the ordinary derivative/anti-derivative $\frac{d^n}{dx^n}$.

This work is organized as follow: In Section 2 we introduce differential–anti-differential equation and then construct the operator with a base function in two independent variables and two parameters and give some examples. The results, in these examples, can be used to solve some partial differential–anti-differential equations including Continuity, Heat, Laplace etc. In Section 3, we take the base function as the function of one independent variable and give some examples in the form of a table. Next, identify the forms of the Homogeneous partial differential–anti-differential equations which can be solved. Several examples including Schrodinger, Euler and Blasius partial differential equations are solved.

2 Construction of the operator, definitions and solution of partial differential–anti-differential equation

In this section first we introduce Homogeneous differential–anti-differential equations and then an operator, with a base function having two independent variables and two parameters, is constructed and some examples are given. Next, we give main results and identify the forms of the partial differential–anti-differential equation which can be solved through the constructed operator. Exact solutions of some well known examples, including Continuity, Heat and Laplace partial differential equations, are also investigated.

An equation which contains derivatives as well as anti-derivatives is called differential–anti-differential equation. The equation

$$F\left(x, y, \frac{\partial U}{\partial x}, \frac{\partial^{-1} U}{\partial x^{-1}}, \frac{\partial U}{\partial y}, \frac{\partial^{-1} U}{\partial y^{-1}}, ..., \frac{\partial^{n+m} U}{\partial x^n \partial y^m}, ...\right) = 0 \tag{2.1}$$

is called the general form of Homogeneous partial differential–anti-differential equation satisfied by a differentiable function U, where n and m are some integers.

An easy method to find the solution of this equations is to convert them into an Auxiliary polynomial equation and then use the roots to get solutions [1, 3, 7].

Construction of the operator D and its use: *Let $U = f(m_1, m_2; x, y)$ is a differentiable base function in two independent variables x and y and two parameters m_1 and m_2. We investigate the operator $D_x(U)$, the first order partial derivative of U with respect to x as follow: First, we calculate*

$$D_x(U) = \lim_{\Delta x \to 0} \frac{f(m_1, m_2; x + \Delta x, y) - f(m_1, m_2; x, y)}{\Delta x}, \tag{2.2}$$

and then implicitly express as $D_x(U) = g(m_1, m_2; x, y, U)$ i.e., a function of m_1, m_2, x, y and U only, and similarly $D_y(U) = h(m_1, m_2; x, y, U)$. Next, using implicit differentiation, we calculate the n^{th} and m^{th} order derivatives $D_{x^n}(U)$, $D_{y^m}(U)$ and $D_{x^n y^m}(U)$ as functions of m_1, m_2, n, m, x, y and U. After this, we replace n and m as integers. Inserting these derivatives in a Homogeneous differential–anti-differential equation (2.1) so that we get the derivative–anti-derivative free equation as

$$F(m_1, m_2, n, m; U) = 0. \tag{2.3}$$

In case $F(m_1, m_2, n, m; U)$ explicitly expressed as

$$F(m_1, m_2, n, m; U) = G(m_1, m_2, n, m)H(n, m, U), \tag{2.4}$$

that is, the product of functions $G(m_1, m_2, n, m)$ and $H(n, m; U)$, then by (2.3) we have

$$G(m_1, m_2, n, m) = 0, \tag{2.5}$$

and this is in fact the Auxiliary or Characteristic equation of (2.1). Next using (2.5), we calculate the roots m_1 *and* m_2 *and insert in the base function* $U = f(m_1, m_2; x, y)$ *to get the exact solution of (2.1).*

Definition 2.1. *The algebraic equation (2.5), the roots of which lead to the solution of the Homogeneous differential–anti-differential equation (2.1), is called an Auxiliary or Characteristic equation.*

Expression in (2.2) shows that $D_x(U)$ is a function of the variables x and U and parameters m_1 and m_2. Next, we come to the following examples:

Example 2.1. *For any* $c \in \mathbb{R}^+$, *let* $U = c^{m_1x+m_2y}$, $m_1, m_2 \in \mathbb{R}$, *be a base function then we can express* $D_x(U) = (m_1 \ln c)c^{m_1x+m_2y} = (m_1 \ln c)U$, $D_y(U) = (m_2 \ln c)U$ *and* $D_{xy}(U) = (m_1m_2 \ln^2 c)U$. *Similarly, the anti-derivatives* $D_{x^{-1}}(U) = (m_1 \ln c)^{-1}U$, $D_{y^{-1}}(U) = (m_2 \ln c)^{-1}U$, $D_{(xy)^{-1}}(U) = (m_1m_2 \ln^2 c)^{-1}U$ *etc. Generally,* $D_{x^ny^m}(U) = (m_1^n m_2^m \ln^{n+m} c)U$, *for any integers* n *and* m.

Let's consider another example as:

Example 2.2. *Let* $U = x^{m_1}y^{m_2}$, *for any natural numbers* m_1, m_2, *and let* $n \le m_1$ *and* $m \le m_2$. *Then one can express* $D_{x^n}(U) = \frac{m_1!}{(m_1-n)!x^n}U$, $D_{y^m}(U) = \frac{m_2!}{(m_2-m)!y^m}U$ *and* $D_{x^ny^m}(U) = \frac{m_1!m_2!}{(m_1-n)!(m_2-m)!x^ny^m}U$, *while the anti-derivatives as* $D_{x^{-n}}(U) = \frac{m_1!x^n}{(m_1+n)!}U$ *and* $D_{y^{-m}}(U) = \frac{m_2!y^m}{(m_2+m)!}U$.

In the above examples, the operators $D_{x^n}(U)$, $D_{y^m}(U)$ and $D_{x^ny^m}(U)$ are functions of n, m, x, y, U and parameters m_1 and m_2.

Next, we show that the operator D can be used to solve some partial differential–anti-differential equations. For this, let us consider the following linear homogeneous partial differential–anti-differential equation, with constant coefficients:

$$\begin{aligned}
&a_n\frac{\partial^n U}{\partial x^n} + a_{n-1}\frac{\partial^n U}{\partial x^{n-1}\partial y} + \dots + a_0\frac{\partial^n U}{\partial y^n} \\
&+b_{n-1}\frac{\partial^{n-1} U}{\partial x^{n-1}} + b_{n-2}\frac{\partial^{n-1} U}{\partial x^{n-2}\partial y} + \dots + b_0\frac{\partial^{n-1} U}{\partial y^{n-1}} \\
&+\dots + c_2\frac{\partial U}{\partial x} + d_2\frac{\partial U}{\partial y} + c_1\frac{\partial^{-1} U}{\partial x^{-1}} + d_1\frac{\partial^{-1} U}{\partial y^{-1}} + eU = 0, \qquad (2.6)
\end{aligned}$$

where n is any integer.

The following result leads the equation (2.6) to the form (2.4).

Theorem 2.3. *For any integers* n *and* m, *the operator* $D_{x^ny^m}(U) = (a^nb^m \ln^{n+m} c)U$ *with base function* $U = c^{ax+by}$, $a, b, c \in \mathbb{R}$ *with* $c > 0$, *converts (2.6) in the form*

$$\left[\ln^n c\,(a_na^n + a_{n-1}a^nb + \dots + a_0b^n) + \dots + \ln c\,(c_2a + d_2b) + \ln^{-1} c\left(\frac{c_1}{a} + \frac{d_1}{b}\right) + e\right]U = 0. \qquad (2.7)$$

Proof. Inserting $D_{x^n y^m}(U) = (a^n b^m \ln^{n+m} c)U$ in (2.6) we get (2.7). $\square$

Next, let's come to the following result:

Theorem 2.4. *If the operator D leads the Homogeneous partial differential–anti-differential (2.1) to condition (2.4) then $U = f(m_1, m_2; x, y)$ is solution of (2.1).*

Proof. If (2.4) is satisfied then, using (2.3), we get $G(m_1, m_2, n, m) = 0$, that is m_1 and m_2 are roots of $G(m_1, m_2, n, m)H(n, m; U) = 0$. As (2.4) is in the explicit form, therefore $U = f(m_1, m_2; x, y)$ is the root of (2.3) and consequently, it is the solution of (2.1). $\square$

The latter result shows if the condition (2.4) holds then $G(m_1, m_2, n, m) = 0$ is the Auxiliary equation of (2.1). In the following, we come to the Auxiliary equation of (2.6).

Corollary 2.5. *Auxiliary equation of (2.6) is*

$$\begin{aligned}&\ln^n c\,(a_n a^n + a_{n-1}a^n b + ... + a_0 b^n) + \ln^{n-1} c\big(b_{n-1}a^{n-1} + b_{n-2}a^{n-2}b + ... + b_0 b^{n-1}\big)\\ &+ ... + \ln c\,(c_1 a + d_1 b) + e = 0. \end{aligned} \tag{2.8}$$

Proof. As the operator in Theorem 2.3 leads the equation (2.6) to condition (2.4) with

$$\begin{aligned} G(a,b) \;=\;& \ln^n c\,(a_n a^n + a_{n-1}a^n b + ... + a_0 b^n) + \ln^{n-1} c\big(b_{n-1}a^{n-1} + b_{n-2}a^{n-2}b\\ &+ ... + b_0 b^{n-1}\big) + ... + \ln c\,(c_1 a + d_1 b) + e. \end{aligned}$$

Thus using (2.5), we get the result. $\square$

Now, we study some applications of our results:

Example 2.6. *Consider the partial differential–anti-differential equation*

$$\frac{\partial U}{\partial x} + \frac{\partial U}{\partial y} + \frac{\partial^{-1} U}{\partial x^{-1}} = 0. \tag{2.9}$$

As this equation is of the form (2.6), so by Corollary 2.5, we get the Auxiliary equation for (2.9) as $a^2 + ab + 1 = 0$. This gives $b = -\frac{1+a^2}{a}$. Using this in the base function $U = e^{ax-by}$, given in Theorem 2.3, we get the solution of (2.9) as $U = e^{ax - \frac{1+a^2}{a}y}$, for any non-zero real number a.

Example 2.7. *Consider the Continuity equation*

$$U_x + U_y = 0. \tag{2.10}$$

By Corollary 2.5, the Auxiliary equation is $a + b = 0$. This gives $a = -b$. Using the base function $U = e^{ax+by}$, we get the solution as $U = e^{a(x-y)}$, for any real number a.

Example 2.8. *Let's come to the well known Heat equation*

$$\frac{\partial U}{\partial t} = k\frac{\partial^2 U}{\partial x^2}. \tag{2.11}$$

Using Corollary 2.5, the Auxiliary equation is $a - kb^2 = 0$. This implies $a = kb^2$. Hence, we get the general solution as $U = e^{kb^2t+bx}$, for any $b \in \mathbb{R}$.

Example 2.9. *Consider the Wave equation*

$$\frac{\partial^2 U}{\partial t^2} = k\frac{\partial^2 U}{\partial x^2}, \tag{2.12}$$

with Auxiliary equation $a^2 - k^2b^2 = 0$. This gives $a = \pm kb$, thus the general solution of (2.12) is $U = e^{\pm kbt+bx}$, for any real number b.

Example 2.10. *Laplace equation is*

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0. \tag{2.13}$$

Using Corollary 2.5, we get $U = e^{a(x\pm iy)}$, for any real number a.

Next, we study non-linear homogeneous partial differential–anti-differential equation as:

$$ax^n\frac{\partial^n U}{\partial x^n} + bx^ry^s\frac{\partial^{r+s}U}{\partial x^r \partial y^s} + cy^m\frac{\partial^m U}{\partial y^m} + dU = 0, \tag{2.14}$$

where a, b, c and d are constants while n, r, s and m are integers.

Lemma 2.11. *The operator $D_{x^ny^m}(U) = \frac{m_1!m_2!}{(m_1-n)!(m_2-m)!x^ny^m}U$ with base function $U = x^{m_1}y^{m_2}$, $m_1, m_2 \in \mathbb{N}$ such that $m_1 \geq n$ and $m_2 \geq m$, converts the equation (2.14) in the form*

$$\left(a\frac{m_1!}{(m_1-n)!} + b\frac{m_1!m_2!}{(m_1-r)!(m_2-s)!} + c\frac{m_2!}{(m_2-m)!} + d\right)U = 0. \tag{2.15}$$

Proof. Inserting $D_{x^ny^m}(U) = \frac{m_1!m_2!}{(m_1-n)!(m_2-m)!x^ny^m}U$ in (2.15) and simplifying we get the result. □

Next, we study the Auxiliary equation of (2.14).

Corollary 2.12. *Auxiliary equation of (2.14) is*

$$a\frac{m_1!}{(m_1-n)!} + b\frac{m_1!m_2!}{(m_1-r)!(m_2-s)!} + c\frac{m_2!}{(m_2-m)!} + d = 0. \tag{2.16}$$

Proof. Lemma 2.11 shows the condition (2.5) does hold with

$$G(m_1, m_2) = a\frac{m_1!}{(m_1 - n)!} + b\frac{m_1!m_2!}{(m_1 - r)!(m_2 - s)!} + c\frac{m_2!}{(m_2 - m)!} + d.$$

Thus, using (2.4), we get the result. □

Let's come to the application of Corollary 2.12.

Example 2.13. *Consider the equation*

$$xU_x - yU_y = 0. \tag{2.17}$$

This equation is of the form (2.14) therefore, using the operator and the base function given in Lemma 2.11, by Corollary 2.11, we get the Auxiliary equation of (2.17) as $m_1 - m_2 = 0$. *This further gives* $m_1 = m_2$. *Inserting in the base function, we get* $U = (xy)^{m_1}$.

3 Solution of Homogeneous ordinary differential–anti-differential equations

In this section, we study the solutions of linear and non-linear ordinary differential–anti-differential equations. First, we give a table on some base functions, their operator and linearity. Next, we identify the types of ordinary differential–anti-differential equation and the corresponding operator which can be used to lead to its Auxiliary equation and then we solve some examples including the well known Schrodinger, Euler and Blasius differential equations.

Let $y = f(m_1; x)$ only, i.e., y is a function of one independent variable x and one parameter m_1, then

$$D(y) = \lim_{h\to 0} \frac{f(m_1; x + h) - f(m_1; x)}{h}. \tag{3.1}$$

Expression (3.1) also shows that $D(y)$ can be implicitly expressed in terms of x, y and m_1. Moreover, the above operator D and its anti-derivative D^{-1} satisfies

$$DD^{-1}(f(x)) = D^{-1}D(f(m_1; x)) = f(m_1; x), \tag{3.2}$$

that is, we assume that $D^{-1}D$ gives only the the principle part $f(m_1; x)$ with no constant due to the anti-derivative. In other words we say the operator anti-derivative may not be equal to the integration operator. This assumption leads to the commutative property $DD^{-1} = D^{-1}D$. One can easily find that the set of derivative–anti-derivatives

$$\{D^n, n \in \mathbb{Z}\} \tag{3.3}$$

forms a commutative group under composition, where $\mathbb{Z}$ is the set of integers and satisfies $D^mD^n = D^{m+n}$ for $m, n \in \mathbb{Z}$.

In the following example, we consider a base function and calculate the operator D.

Example 3.1. *Let $y = ce^{mx}$ be a base function where m and c are any number, then through implicit differentiation/anti-differentiation, we get the n^{th} order derivative/anti-derivative as $D^n(y) = m^n y$.*

The following table studies the n^{th} order derivative operator $D^n(y)$, $n \in \mathbb{N}$, for few base functions $y = f(m; x)$. Note that similar expression for the anti-derivative, i.e., D^{-n}, $n \in \mathbb{N}$, can be calculated easily.

Base Function y	$D(y)$	$D^n(y)$	Linearity of $D^n(y)$
m^x	$(\ln m)\, y$	$(\ln^n m)\, y$	linear
$e^{\lambda x}$	λy	$\lambda^n y$	linear
x^m	$\frac{m}{x} y$	$\frac{\Gamma(m+1)}{\Gamma(m-n+1)x^r} y$	nonlinear
$\ln(mx)$	me^{-y}	$(-1)^{r-1} m^n (n-1)! e^{-ny}$, $r > 1$	nonlinear
$-\frac{1}{mx}$	my^2	$m^n n! y^{n+1}$	nonlinear
$sin(mx)$	$\pm m\sqrt{1-y^2}$	$(-1)^{\frac{n}{2}} m^n y$, n is even	linear
$sin(mx)$	$\pm m\sqrt{1-y^2}$	$\pm(-1)^{\frac{n+3}{2}} m^n \sqrt{1-y^2}$, n is odd	nonlinear
$sinh(mx)$	$\pm m\sqrt{1+y^2}$	$m^n y$, n is even	linear
$sinh(mx)$	$\pm m\sqrt{1+y^2}$	$\pm m^n \sqrt{y^2+1}$, n is odd	nonlinear

The operator $D^n(y)$, $n \in \mathbb{N}$, for few base functions y.

Key Point:

1. In the above table one can see that each $D^n(y)$ can be explicitly or implicitly expressed as the function of y and this is the key point which leads to the solution of any Homogeneous differential–anti-differential equation.

2. If $D^n(y)$ does not depend on the independent variable x then it can be used to solve linear differential–anti-differential equations with constant coefficients and if it depends on the dependent variable then it can be used to solve some non-linear differential–anti-differential equations with variable coefficients.

Now, we are ready to study the solution process of the Homogeneous differential–anti-differential equations with constant coefficients of the form

$$a_n \frac{d^n y}{dx^n} + a_{n-1} \frac{d^{n-1} y}{dx^{n-1}} + ... + a_1 \frac{dy}{dx} + a_0 y = 0, \tag{3.4}$$

where n is any integer.

Corollary 3.2. *If $D^n(y) = (\ln^n m)\, y$ with base function $y = m^x$, for any integer n, then Auxiliary equation of (3.4) is*

$$a_n \ln^n m + a_{n-1} \ln^{n-1} m + ... + a_1 \ln m + a_0 = 0. \tag{3.5}$$

Proof. Inserting $D^n(y)$ in (3.4), we get

$$\big(a_n \ln^n m + a_{n-1} \ln^{n-1} m + ... + a_1 \ln m + a_0\big)\, y = 0. \tag{3.6}$$

Putting the coefficient of y in (3.6) equal to zero we get (3.5). □

Corollary 3.3. *If $D^n(y) = \lambda^n y$, for any natural number n, then the Auxiliary equation of (3.4) is*

$$a_n \lambda^n + a_{n-1} \lambda^{n-1} + ... + a_1 \lambda + a_0 = 0. \tag{3.7}$$

Proof. Replacing $\ln^n m$ by λ^n in (3.5), we get the result. □

In the following, we study some applications of Corollary 3.3.

Theorem 3.4. *If the operator $D(y)$ is linear and of the form $D(y) = my$, for any real number m, then there exist i distinct number of polynomials of the form*

$$a_n m_i^n + a_{n-1} m_i^{n-1} + ... + a_1 m_i + a_0 = 0, \tag{3.8}$$

for $0 \leq i \leq n$, with $D(y) = m_i y$. Moreover, the set $\{y : D(y) = m_i y\}$ is linearly independent.

Proof. As $D(y) = my$ is linear thus $D^n(y) = m^n y$, for any positive integer n. Moreover by Corollary 3.3, m satisfies a polynomial of the form (3.7) having i, $0 \leq i \leq n$, number of roots. In other words m has i number of values say m_i satisfying $D(y) = m_i y$. Using again (3.7), we get a system of i number of equations of the form (3.8).

For the second part, as the system of equations (3.8) is Homogeneous so the set $\{m_i\}$ forms a basis for the solution space and hence, $\{y : D(y) = m_i y\}$ is linearly independent. □

In the following some linear Homogeneous differential–anti-differential equations with constant coefficients are solved.

Example 3.5. *Consider the differential–anti-differential equation*

$$2\frac{dy}{dx} + 5y + 3\frac{d^{-1}y}{dx^{-1}} = 0. \tag{3.9}$$

This equation is of the form (3.4), by Corollary 3.3, the Auxiliary equation of (3.9) is

$$2m + 5 + \frac{3}{m} = 0,$$

which gives $m = -1$ *and* $-\frac{3}{2}$. *Inserting in the base function* $y = e^{mx}$, *we get* $y = ce^{-x} + de^{-\frac{3}{2}x}$.

Since, through our constructed operator, more than one Auxiliary equations can be found. To show this, let's come to the following example:

Example 3.6. *Let us solve*

$$\frac{d^2y}{dx^2} + 7\frac{dy}{dx} + 12y = 0. \tag{3.10}$$

The given equation is of the form (3.4), then by Corollary 3.2, the Auxiliary equation, through $D(y) = (\ln m)y$, *is*

$$\ln^2 m + 7\ln m + 12 = 0, \tag{3.11}$$

which gives $\ln m = -3$, *and* -4 *that is,* $m = e^{-3}$ *and* e^{-4}. *Using the base function* $y = m^x$, *we get* $y = ce^{-3x} + de^{-4x}$.

If we use the operator as $D(y) = my$ *with base function* $y = e^{mx}$, *then another form of the Auxiliary equation is*

$$m^2 + 7m + 12 = 0, \tag{3.12}$$

with roots $m = -3, -4$. *Roots lead to the same solution* $y = ce^{-3x} + de^{-4x}$.

In the the next well-known example, we can use the operator $D(y)$ having several forms my, $\ln|m|$, $m\sqrt{1-y^2}$, $m\sqrt{1+y^2}$ etc.

Example 3.7. *Consider the Schrodinger differential equation*

$$\frac{d^2y}{dx^2} + ky = 0, \tag{3.13}$$

where k *is Schrodinger constant.*

Let's use $D(y) = my$ *with base function* $y = e^{mx}$ *in (3.13) we get*

$$m^2 + k^2 = 0, \tag{3.14}$$

with solutions $m = -ki$ *and* ki. *Thus the general solutions of Schrodinger differential equation (3.13) is* $y = c_1e^{-kix} + c_2e^{kix}$.

Example 3.8. *Consider the differential equation*

$$\frac{d^3y}{dx^3} - 9\frac{dy}{dx} = 0. \tag{3.15}$$

Since all the coefficients are constant, we can choose the base functions $y = e^{mx}$, $y = m^x$, $y = sin(mx)$, $cos(mx)$, $sinh(mx)$ and $cosh(mx)$, to solve (3.15).

In the next, we come to the solution of linear homogeneous differential–anti-differential equations with variable coefficients of the form:

$$a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1(x)\frac{dy}{dx} + a_0(x)y = 0. \tag{3.16}$$

When $a_k(x) = c_kx^k, k = 0, 1, ..., n$, then (3.16) is called Euler's ordinary differential–anti-differential equation of order n.

We find that some base functions have potential to lead to the Auxiliary equation of (3.16). For this, we come to the following result:

Lemma 3.9. *The operator $D^n(y) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)x^n}y$ with $y = x^m$, m is any constant, converts Euler differential–anti-differential equations of the form (3.16) with $a_k(x) = c_kx^k, k = 0, 1, ..., n$, to the form (2.4), where*

$$G(m) = c_n\frac{\Gamma(m+1)}{\Gamma(m-n+1)} + c_{n-1}\frac{\Gamma(m+1)}{\Gamma(m-n+2)} + ... + c_1m + c_0. \tag{3.17}$$

Proof. Using the given $D^n(y)$ in (3.16), we get

$$c_nx^n\frac{\Gamma(m+1)}{\Gamma(m-n+1)x^{n-1}}y + c_{n-1}x^{n-1}\frac{\Gamma(m+1)}{\Gamma(m-n+2)x^{n-1}}y + ... + c_1x\frac{m}{x}y + c_0y = 0. \tag{3.18}$$

Simplifying (3.18), we get the result. □

Example 3.10. *Let we are given*

$$x^2\frac{d^2y}{dx^2} + 5x\frac{dy}{dx} + 4y = 0. \tag{3.19}$$

As (3.19) is an Euler equation, therefore we use $D^n(y) = \frac{\Gamma(m+1)}{\Gamma(m-n+1)x^n}y$ with base function $y = x^m$. Using Lemma 3.9 we get the Auxiliary equation as

$$f(m) = m^2 + 4m + 4 = 0, \tag{3.20}$$

with root $m = -2$ having multiplicity 2. Thus, we get a general solution of (3.19) as

$$y = d_1x^{-2} + d_2x^{-2}lnx,$$

where d_1 and d_2 are constants.

Next, we discuss non-linear ordinary homogeneous differential equations of the form

$$a_n b^{(n+1)y}\frac{d^n y}{dx^n} + ... + a_1 b^{2y}\frac{dy}{dx} + a_0 b^y = 0, \tag{3.21}$$

where a_i are arbitrary constants, b is any positive real number and n is any natural number.

Lemma 3.11. *The operator $D^n(y) = \frac{(-1)^{n+1}m^n(n-1)!}{\ln b}b^{-ny}$, $b > 0$ and $n \in \mathbb{N}$, with base function $y = \log_b(mx)$ leads (3.21) to*

$$\frac{1}{\ln b}\left[a_n(-1)^{n+1}m^n(n-1)! + a_{n-1}(-1)^n m^{n-1}(n-2)! + ... + a_1 m + a_0\right] b^y = 0. \tag{3.22}$$

Proof. Inserting $D^n(y) = \frac{(-1)^{n+1}m^n(n-1)!}{\ln b}b^{-ny}$ in (3.21) and simplify we get the result. □

Corollary 3.12. *Auxiliary equation of (3.21) is*

$$a_n(-1)^{n+1}m^n(n-1)! + a_{n-1}(-1)^n m^{n-1}(n-2)! + ... + a_1 m + a_0 = 0. \tag{3.23}$$

Proof. Using Lemma 3.11, we get the result. □

Example 3.13. *As the differential equation*

$$e^{3y}\frac{d^2y}{dx^2} + 4e^{2y}\frac{dy}{dx} - 3e^y = 0 \tag{3.24}$$

is of the form (3.21), therefore using Corollary 3.12, we get the Auxiliary equation as

$$-m^2 + 4m - 3 = 0 \tag{3.25}$$

with roots $m = 1$ and 3. Using $y = \log_b(mx)$, we get $y = c_1\ln(x) + c_2\ln(3x)$.

After this, we study non-linear homogeneous differential equations of the form

$$a_n\frac{d^n y}{dx^n} + a_{n-1}y\frac{d^{n-1}y}{dx^{n-1}} + ... + a_1 y^{n-1}\frac{dy}{dx} + a_0 y^n = 0, \tag{3.26}$$

where a_i are arbitrary constants while n is any natural number.

Lemma 3.14. *The operator $D^n(y) = m^n n! y^{n+1}$ with $y = -\frac{1}{mx}$, $m \neq 0$, and n is a natural number, leads the Homogeneous differential equations of the form (3.26) to*

$$\left(a_n m^n n! + a_{n-1}m^{n-1}(n-1)! + ... + a_1 m + a_0\right) y^{n+1} = 0. \tag{3.27}$$

Proof. Using $D^n(y) = m^n n! y^{n+1}$ in (3.26) and simplifying, we get the result. □

Corollary 3.15. *Auxiliary equation of the differential equation (3.26) is*

$$a_n m^n n! + a_{n-1} m^{n-1}(n-1)! + ... + a_1 m + a_0 = 0. \tag{3.28}$$

Proof. Using Lemma 3.14, we get the result. □

Example 3.16. *Let we are given the Blasius equation as*

$$\frac{d^3y}{dx^3} + y\frac{d^2y}{dx^2} = 0. \tag{3.29}$$

As (3.29) is of the form (3.26), therefore using Corollary 3.15 we obtain

$$6m^3 + 2m^2 = 0. \tag{3.30}$$

The only non-zero root of (3.30) is $m = -\frac{1}{3}$. *Using* $y = -\frac{1}{mx}$, *we get* $y = \frac{3c_1}{x}$.

Conclusion

We have introduced an idea of ordinary and partial differential–anti-differential equations and developed a simple and short method to find the exact solution of several types of ordinary and partial differential–anti-differential equations. Method can be used to solve several types of well-known ordinary, partial as well as fractional differential–anti-differential equations.

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