EXISTENCE AND STABILITY OF POSITIVE SOLUTIONS FOR A HADAMARD-TYPE FRACTIONAL TWO-POINT BOUNDARY VALUE PROBLEM

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Abstract. In this paper,we mainly establish existence and uniqueness of positive solution for the Hadamard-type fractional two-point boundary value problem

$$\begin{cases} (\mathcal{D}^{\alpha}_{1^+}x(t))'' + \lambda^2\mathcal{D}^{\alpha}_{1^+}x(t) + f(t, x(t), -\mathcal{D}^{\alpha}_{1^+}x(t)) = 0, \ t \in (1, e), \\ x(1) = x'(1) = x'(e) = 0, \mathcal{D}^{\alpha+1}_{1^+}x(1) = \mathcal{D}^{\alpha+1}_{1^+}x(e) = 0, \end{cases}$$

by using the fixed point theorems. In addition, we also study the Ulam-Hyers-Rassias stability of the related problem. On the other hand, when $f(t, u, v)$ is singular at $u = 0$ and $v = 0$, we study the existence and uniqueness of its solution. Finally, some examples are included to show the applicability of our results.

1. Introduction

During the last two decades, many real world processes and phenomena in signal processing, control theory, fluid flow, electrical systems, etc., are characterized by fractional-order differential equations(FDEs), which is regarded as better and improved over ones depending only on integer-order operators, see [11, 12, 21]. Therefore, the existence, uniqueness and stability of the solutions are well studied recently for the fractional differential equations of Riemann-Liouville or Caputo form. We refer the readers to [2, 3, 4, 7, 14, 16, 17, 22, 24, 25, 26, 27, 28] and the references therein. In addition, multiple researchers have applied various methods from fixed point theory and variational theory to the fractional p-Laplacian problems, see [10, 18, 19, 20] and the references therein. In 1892, another known type of fractional derivatives defined in the literature is the fractional derivative duo to Hadamard [13], varying from the aforementioned derivatives in the sense that the integral kernel in the Hadamard derivative description contains an arbitrary exponents logarithmic function. Since then, it is important to study the property and application of Hadamard-type initial and boundary value problems, see [1, 5, 6, 8, 9, 15, 23]. Especially, the authors [29] has applied the nonlinear analysis methods combining with some numerical techniques to study the existence of positive solutions for a class of weakly singular Hadamard-type fractional mixed periodic boundary value problems with a changing-sign singular perturbation.

$$\begin{cases} -(\mathcal{D}^{\beta}_{t}z(t))'' - b(t)\mathcal{D}^{\beta}_{t}z(t) = f(t, z(t), -\mathcal{D}^{\beta}_{t}z(t)) + \chi(t), \ t \in (1, e), \\ z(1) = z'(1) = z'(e) = 0, \mathcal{D}^{\beta}_{t}z(1) = \mathcal{D}^{\beta}_{t}z(e), \mathcal{D}^{\beta+1}_{t}z(1) = \mathcal{D}^{\beta+1}_{t}z(e). \end{cases}$$

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where $2 < \beta \leq 3$, $\mathcal{D}_t^\beta$ is the Hadamard fractional derivative of order α, β, $b \in L^p(1,e)$. The nonlinearity can be singular at the second and third variables and be changing-sign.

Inspired by these references, we are interested in studying the following Hadamard fractional two-point boundary value problem

$$\begin{cases} (\mathcal{D}_{1^+}^\alpha x(t))'' + \lambda^2 \mathcal{D}_{1^+}^\alpha x(t) + f(t, x(t), -\mathcal{D}_{1^+}^\alpha x(t)) = 0, \ t \in (1,e), \\ x(1) = x'(1) = x'(e) = 0, \mathcal{D}_{1^+}^{\alpha+1} x(1) = \mathcal{D}_{1^+}^{\alpha+1} x(e) = 0, \end{cases} \tag{1.1}$$

where $2 < \alpha \leq 3$ and $0 < \lambda \leq \frac{1}{\sqrt{e(e-1)}}$. We will study the existence, uniqueness and stability of positive solution to this equation by Banach's contraction mapping principle and Schauder's fixed point theorem when f is continuous. On the other hand, we study the existence and uniqueness of positive solution to this equation by fixed point theorem for mixed monotone operators when $f(t,u,v)$ is singular at $u = 0$ and $v = 0$. Compared to results in the references, our work presented in this paper has some new features. Firstly, the problem satisfies the Neumann boundary condition, which is obviously different to the periodic boundary value condition. Secondly, we applied the iterative method to obtain some sufficient conditions for the unique existence result. Finally, we study the Ulam-Hyers stability result.

The present paper is organized as follows. In Section 2, we introduce the notations of the Hadamard fractional integral and differential operator, and the Green functions of two linear differential equations. In Section 3, the existence and uniqueness results of the solution of the problem (1.1) are presented. In Section 4, the Ulam-Hyers-Rassias stability result is given. In Section 5, some examples are given to illustrate our main results.

2. Preliminaries

In this section, we will use the following definitions for Hadamard-type fractional integrals and derivatives.

Definition 2.1.[16] The Hadamard integral of fractional order $\alpha > 0$ for a function $x : [a, +\infty) \to \mathbb{R}$ is defined by

$$I_{a^+}^\alpha x(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (\ln \frac{t}{s})^{\alpha-1} x(s) \frac{ds}{s}$$

provided the right-hand side is pointwise defined on $(a, +\infty)$.

Definition 2.2.[16] The Hadamard derivative of fractional order $\alpha > 0$ for a function $x : [a, +\infty) \to \mathbb{R}$ is defined by

$$\mathcal{D}_{a^+}^\alpha x(t) = \frac{1}{\Gamma(n-\alpha)} (t \frac{d}{dt})^n \int_a^t (\ln \frac{t}{s})^{n-\alpha-1} x(s) \frac{ds}{s},$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the largest integer which is less than or equal to α.

Lemma 2.3.[16] Assume that $\alpha > 0$, then

$$I_{a^+}^\alpha \mathcal{D}_{a^+}^\alpha x(t) = x(t) + \sum_{i=1}^n C_i t^{\alpha-i},$$

for some $C_i \in \mathbb{R}$, $i = 1, 2, ..., n$, where n is the smallest integer greater than or equal to α.

Now, let $-\mathcal{D}^{\alpha}_{1^+}x(t) = u(t)$ with $x(1) = x'(1) = x'(e) = 0$. From [29] it follows that

$$x(t) = \int_1^e G_1(t, s)u(s)\frac{ds}{s},$$

where

$$G_1(t, s) = \frac{1}{\Gamma(\alpha)}\begin{cases}(\ln t)^{\alpha-1}(1 - \ln s)^{\alpha-2} - (\ln t - \ln s)^{\alpha-1}, & 1 \le s \le t \le e,\\ (\ln t)^{\alpha-1}(1 - \ln s)^{\alpha-2}, & 1 \le t \le s \le e.\end{cases}$$

Then the problem (1.1) can be reduced to the following Neumann boundary value problem:

$$\begin{cases} u''(t) + \lambda^2 u(t) = f(t, \int_1^e G_1(t, s)u(s)\frac{ds}{s}, u(t)), \ t \in (1, e),\\ u'(1) = u'(e) = 0.\end{cases} \tag{2.1}$$

Furthermore, let $G_2(t, s)$ be the Green function of the homogeneous linear problem

$$\begin{cases} u''(t) + \lambda^2 u(t) = 0,\\ u'(1) = u'(e) = 0,\end{cases} \tag{2.2}$$

that is

$$G_2(t, s) = \frac{1}{\lambda \sin \lambda(e - 1)}\begin{cases}\cos \lambda(1 - t) \cos \lambda(s - e) + \sin \lambda(e - 1) \sin \lambda(t - s), & 1 \le s \le t \le e,\\ \cos \lambda(1 - t) \cos \lambda(s - e), & 1 \le t \le s \le e.\end{cases}$$

Finally, we give the properties of the Green's functions as follows.

Lemma 2.4. The Green's functions $G_i(t, s)(i = 1, 2)$ satisfy the following properties:
(1) $G_i(t, s) \ge 0$;
(2) For all $t, s \in [1, e]$, the following inequalities hold:

- *(i)* $(\ln t)^{\alpha-1}G_1(e, s) \le G_1(t, s) \le \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}$;
- *(ii)* $G_2(s, s) \le G_2(t, s) \le \frac{\cos \lambda(e-t)}{\lambda \sin \lambda(e-1)}$.

Proof. From the expression of $G_i(t, s)$, it is obvious that property (1) is valid. Now we prove property (2).

(i) For $1 \le s \le t \le e$,

$$\begin{aligned} G_1(t, s) &= \frac{1}{\Gamma(\alpha)}\Big((\ln t)^{\alpha-1}(1 - \ln s)^{\alpha-2} - (\ln t - \ln s)^{\alpha-1}\Big)\\ &= \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\Big((1 - \ln s)^{\alpha-2} - (1 - \frac{\ln s}{\ln t})^{\alpha-1}\Big)\\ &\ge \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}\ln s(1 - \ln s)^{\alpha-2}\\ &= (\ln t)^{\alpha-1}G_1(e, s).\end{aligned}$$

The same is true for $1 \le t \le s \le e$, namely $G_1(t, s) \ge (\ln t)^{\alpha-1}G_1(e, s)$ for $t, s \in [1, e]$. Therefore

$$G_1(t, s) \ge (\ln t)^{\alpha-1}G_1(e, s).$$

On the other hand, for $1 \le s \le t \le e$,

$$\begin{aligned} G_1(t,s) &= \frac{1}{\Gamma(\alpha)}\Big[(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-2} - (\ln t - \ln s)^{\alpha-1}\Big] \\ &\le \frac{1}{\Gamma(\alpha)}(\ln t)^{\alpha-1}(1-\ln s)^{\alpha-2} \\ &\le \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}. \end{aligned}$$

For $1 \le t \le s \le e$, it is easily visible for $G_1(t,s) \le \frac{(\ln t)^{\alpha-1}}{\Gamma(\alpha)}$. (i) is hold.

(ii) According to the continuity of $G_2(t,s)$, taking the derivative of $G_2(t,s)$ with respect to t, we have:
for $1 \le s \le t \le e$,

$$\frac{\partial G_2(t,s)}{\partial t} = \frac{1}{\sin\lambda(e-1)}[\cos\lambda(e-s)\sin\lambda(1-t) + \sin\lambda(e-1)\cos\lambda(t-s)]$$

and

$$\frac{\partial^2 G_2(t,s)}{\partial t^2} = \frac{\lambda}{\sin\lambda(e-1)}[-\cos\lambda(e-s)\cos\lambda(1-t) - \sin\lambda(e-1)\sin\lambda(t-s)] \le 0,$$

so $\frac{\partial G_2(t,s)}{\partial t} \ge \frac{\partial G_2(e,s)}{\partial t} = \frac{\sin\lambda(s-1)}{\sin\lambda(e-1)} \ge 0$;
for $1 \le t \le s \le e$,

$$\frac{\partial G_2(t,s)}{\partial t} = \frac{1}{\sin\lambda(e-1)}\cos\lambda(e-s)\sin\lambda(1-t) \le 0.$$

Therefore $G_2(t,s)$ is decreasing with respect to t on $[1,s]$ and increasing with respect to t on $[s,e]$, which means $G_2(t,s) \ge G_2(s,s)$ for $t,s \in [1,e]$.

In addition, for $1 \le t \le s \le e$, from $0 < \lambda \le \frac{1}{\sqrt{e(e-1)}} < \frac{\pi}{2(e-1)}$, we know that $0 < \cos\lambda(s-e) \le 1$, so

$$G_2(t,s) = \frac{1}{\lambda\sin\lambda(e-1)}\cos\lambda(1-t)\cos\lambda(s-e) \le \frac{\cos\lambda(1-t)}{\lambda\sin\lambda(e-1)}.$$

For $1 \le s \le t \le e$, taking the derivative of $G_2(t,s)$ with respect to s, similar to the discussion above, $\frac{\partial G_2(t,s)}{\partial s}$ is monotonically decreasing with respect to s on $[1,t]$. Since $\frac{\partial G_2(t,1)}{\partial s} = 0$, $G_2(t,s)$ is decreasing with respect to s on $[1,t]$, namely,

$$G_2(t,s) \le G_2(t,1) = \frac{\cos\lambda(e-t)}{\lambda\sin\lambda(e-1)}.$$

Hence,

$$G_2(s,s) \le G_2(t,s) \le \frac{\cos\lambda(e-t)}{\lambda\sin\lambda(e-1)},$$

which completes the proof.$\diamond$

3. Existence

For convenience, in this part, let us remember the Banach space $E = C[1,e]$ is a continuous real-valued functions defined on $[1,e]$ endowed with the norm $\|x\| := \max_{t\in[1,e]} |x(t)|$.

Theorem 3.1. Assume that $f : [1,e]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}^+$ is continuous. In addition, f satisfies the following assumptions:
$(H1)$ there exists two positive constants $\mathcal{L}_1, \mathcal{L}_2 > 0$ such that

$$|f(t,x,u) - f(t,y,v)| \le \mathcal{L}_1|x-y| + \mathcal{L}_2|u-v|, \forall\, x,y,u,v \in E, t\in[1,e].$$

Then the Hadamard-type fractional differential equation (1.1) has a unique solution, if $(\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2)\frac{(e-1)}{\lambda\sin\lambda(e-1)} < 1$.

Proof. First, we let $C = \sup_{t\in[1,e]} f(t,0,0)$. Define the operator

$$Tu(t) = \int_1^e G_2(t,s) f(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}, u(s))ds.$$

Now we show that $T : B_r \to B_r$ and T is a contraction map, where $B_r = \{u\in E : \|u\| < r\}$ with $r > \frac{C\Gamma(\alpha)(e-1)}{\lambda\Gamma(\alpha)\sin\lambda(e-1) - (\mathcal{L}_1 + \mathcal{L}_2\Gamma(\alpha))(e-1)}$.

On one hand, for any $u \in B_r$, from Lemma 2.4 and (H1), we have

$$\begin{aligned}
\|Tu(t)\| &= \|\int_1^e G_2(t,s) f(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}, u(s))ds\| \\
&\le \int_1^e G_2(t,s)[|f(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}, u(s)) - f(s,0,0)| + C]ds \\
&\le [(\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2)\|u\| + C]\int_1^e G_2(t,s)ds \\
&\le [(\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2)\|u\| + C]\frac{e-1}{\lambda\sin\lambda(e-1)} \\
&< r,
\end{aligned}$$

which means that $T(B_r) \subset B_r$.

On the other hand, for any $u, v \in E$, we have

$$\begin{aligned}
\|Tu(t) - Tv(t)\| &= \|\int_1^e G_2(t,s)\Big(f(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}, u(s)) - f(s, \int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}, v(s))\Big)ds\| \\
&\le \int_1^e G_2(t,s)\Big(\mathcal{L}_1\int_1^e G_1(s,\tau)|u(\tau) - v(\tau)|\frac{d\tau}{\tau} + \mathcal{L}_2|u(s) - v(s)|\Big)ds \\
&\le (\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2)\|u - v\|\int_1^e G_2(t,s)ds \\
&\le (\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2)\frac{e-1}{\lambda\sin\lambda(e-1)}\|u - v\|,
\end{aligned}$$

which implies that T is a contraction map.

Therefore, by the Banach's contraction mapping principle, it follows that the operator T has a unique fixed point, which is the unique solution for the problem (2.1), i.e., the problem (1.1) has the unique solution.$\diamond$

Let

$$P = \{u \in C[1,e] : u(t) \ge 0\},$$

and

$$P_0 = \{u \in P : l \leq u(t) \leq L\}.$$

Obviously, P_0 is a subset of P and $P \subseteq E$.

Theorem 3.2. Suppose that $f : [1, e] \times \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is continuous. In addition, f satisfies the following assumptions:
$(H2)$ There exist a constant $\gamma \in (0, 1)$ and two functions $h_1, h_2 \in P$ with $h_1, h_2 \neq 0$ on any subinterval of $(1, e)$, such that

$$h_1(t)(u + v)^\gamma \leq f(t, u, v) \leq h_2(t)(u + v)^\gamma,\ (u, v) \in [0, \infty) \times [0, \infty), t \in [1, e].$$

Then the Hadamard-type fractional differential equation (1.1) has at least one positive solution.

Proof. Defined as the operator in Theorem 3.1, from the above discussion, we can see that the solution of equation (1.1) can be equivalent to the fixed point of the operator $Tu = u$ on P_0.

By Lemma 2.4, for any $u \in P_0$, we get

$$\frac{l(\ln s)^{\alpha-1}}{\alpha(\alpha-1)\Gamma(\alpha)} \leq \int_1^e G_1(s, \tau)u(\tau)\frac{d\tau}{\tau} \leq \frac{L(\ln s)^{\alpha-1}}{\Gamma(\alpha)} \leq \frac{L}{\Gamma(\alpha)}. \tag{3.1}$$

From (3.1) and (H2), we have

$$h_1(s)l^\gamma \leq f(s, \int_1^e G_1(s, \tau)u(\tau)\frac{d\tau}{\tau}, u(s)) \leq h_2(s)L^\gamma\Big(1 + \frac{1}{\Gamma(\alpha)}\Big)^\gamma.$$

Let

$$M_1 = \int_1^e G_2(s, s)h_1(s)ds,\ M_2 = \frac{(1 + \Gamma(\alpha))^\gamma}{(\Gamma(\alpha))^\gamma \lambda \sin \lambda(e-1)} \int_1^e h_2(s)ds.$$

Therefore, combining Lemma 2.4, we have

$$\begin{aligned} Tu(t) &\geq l^\gamma \int_1^e G_2(t, s)h_1(s)ds \\ &\geq l^\gamma \int_1^e G_2(s, s)h_1(s)ds \\ &= M_1 l^\gamma, \end{aligned}$$

and

$$\begin{aligned} Tu(t) &\leq L^\gamma\Big(1 + \frac{1}{\Gamma(\alpha)}\Big)^\gamma \int_1^e G_2(t, s)h_2(s)ds \\ &\leq \frac{L^\gamma\Big(1 + \frac{1}{\Gamma(\alpha)}\Big)^\gamma}{\lambda \sin \lambda(e-1)} \int_1^e h_2(s)ds \\ &= M_2 L^\gamma. \end{aligned}$$

Next, we shall choose a suitable $L > 0$ such that $T : P_0 \to P_0$. In order to get this, we only need to choose $0 < l < L$ such that $M_1 l^\gamma \geq l$ and $M_2 L^\gamma \leq L$. In fact, take $l = \frac{1}{L}$ and then it follows from $M_1 > 0$ and $0 < \gamma < 1$ that there exists a sufficiently large $L > 1$ such that $M_1 L^{1-\gamma} \geq 1$ and $M_2 L^{\gamma-1} \leq 1$.

Thus, from Schauder's fixed point theorem, T has a fixed point $u \in P_0$, and hence, the Hadamard-type fractional differential equation (1.1) has at least one positive solution:

$$x(t) = \int_1^e G_1(t,s)u(s)\frac{ds}{s}.$$

Therefore, the Theorem 3.2 is proved.$\diamond$

When $f(t,u,v)$ is singular at $u = 0$ and $v = 0$, in order to obtain the existence and uniqueness of the solution of Equation (1.1), we first consider the following equation:

$$\begin{cases} u''(t) + \lambda^2 u(t) = f(t, \int_1^e G_1(t,s)u(s)\frac{ds}{s} + \frac{1}{n}, u(t) + \frac{1}{n}),\ t \in (1,e), \\ u'(1) = u'(e) = 0, \end{cases} \tag{3.2}$$

where $n \in N^+$ with $n \geq 2$. Assume that $f : [1,e] \times (0,\infty)^2 \to [0,\infty)$ is continuous, similar to the discussion in the previous section, u is a solution to Equation (3.2) if and only if u is a solution to the following integral equation:

$$u(t) = \int_1^e G_2(t,s)f(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau} + \frac{1}{n}, u(s) + \frac{1}{n})ds.$$

Theorem 3.3. Assume that:
$(H3)$ $f(t,u,v) = \varphi(t,u,v) + \psi(t,u,v)$, where $\varphi : [1,e] \times [0,+\infty) \times [0,+\infty) \to [0,+\infty)$, $\psi : [1,e] \times (0,+\infty) \times (0,+\infty) \to [0,+\infty)$ are continuous, and $\varphi(t,u,v)$ is non-decreasing and $\psi(t,u,v)$ is nonincreasing in $u, v > 0$ respectively.
$(H4)$ there exists $0 < \sigma < \frac{1}{2}$ such that, for $u, v > 0$ and any $c \in (0,1)$,

$$\varphi(t,cu,cv) \geq c^\sigma \varphi(t,u,v), \psi(t,c^{-1}u,c^{-1}v) \geq c^\sigma \psi(t,u,v).$$

Then the Hadamard-type fractional differential equation (1.1) has a unique positive solution x^*, and there exists a constant $0 < M < 1$ such that

$$M \cos\lambda(e-t) \leq x^*(t) \leq \frac{1}{M}\cos\lambda(e-t).$$

Moreover, for any initial u_0, $v_0 \in \tilde{P}_0$, we can construct successively two sequences by

$$u_n = \int_1^e G_2(t,s)\Big[\varphi(s, \int_1^e G_1(s,\tau)u_{n-1}(\tau)\frac{d\tau}{\tau}, u_{n-1}(s)) + \psi(s, \int_1^e G_1(s,\tau)v_{n-1}(\tau)\frac{d\tau}{\tau}, v_{n-1}(s))\Big]ds, n = 1,2,...,$$

$$v_n = \int_1^e G_2(t,s)\Big[\varphi(s, \int_1^e G_1(s,\tau)v_{n-1}(\tau)\frac{d\tau}{\tau}, v_{n-1}(s)) + \psi(s, \int_1^e G_1(s,\tau)u_{n-1}(\tau)\frac{d\tau}{\tau}, u_{n-1}(s))\Big]ds, n = 1,2,...,$$

and the iterative sequences $u_n(t)$, $v_n(t)$ converge uniformly to $-\mathcal{D}_{1^+}^\alpha x^*(t)$ on $[1,e]$ as $n \to \infty$, i.e.,

$$u_n \to -\mathcal{D}_{1^+}^\alpha x^*, \quad v_n \to -\mathcal{D}_{1^+}^\alpha x^* \ \ as \ \ n \to \infty.$$

Remark 3.4. By (H4), for $c \geq 1$, and $u, v > 0$, one has

$$\varphi(t,cu,cv) \leq c^\sigma \varphi(t,u,v), \psi(t,c^{-1}u,c^{-1}v) \leq c^\sigma \psi(t,u,v).$$

In order to prove Theorem 3.3, we first give the necessary definition and Lemma.

Definition 3.5.[28] Assume P to be a normal cone of a Banach space E, $T : P \times P \to P$ is said to be mixed monotone if $T(x,y)$ is non-decreasing in x and non-increasing in y, i.e., $x_1 \leq x_2$ $(x_1, x_2 \in P)$ implies $T(x_1,y) \leq T(x_2,y)$ for any $y \in P$, and $y_1 \leq y_2$ $(y_1, y_2 \in P)$ implies $T(x,y_1) \geq T(x,y_2)$ for any $x \in P$. The element $x^* \in P$ is called a fixed point of T if $T(x^*,x^*) = x^*$.

Lemma 3.6.[28] Let P be a normal, solid cone of Banach space E, and $T:\tilde{P_0}\to\tilde{P_0}$ be a mixed monotone operator. Suppose that there exists $0<\sigma<1$ such that

$$T(cx,c^{-1}y)\geq c^{\sigma}T(x,y),\quad x,y\in\tilde{P_0},0<c<1. \tag{3.3}$$

Then the operator T has a unique fixed point $x^*\in\tilde{P_0}$. Moreover, for any initial x_0, $y_0\in\tilde{P_0}$, by constructing successively the sequences

$$x_n=T(x_{n-1},y_{n-1}),\qquad y_n=T(y_{n-1},x_{n-1}),n=1,2,...,$$

we have $\|x_n-x^*\|\to 0$ and $\|y_n-x^*\|\to 0$ as $n\to+\infty$.

Proof of Theorem 3.3. Let

$$\tilde{P_0}=\{u\in E:M\cos\lambda(e-t)\leq u(t)\leq\frac{1}{M}\cos\lambda(e-t),\ M\in(0,1)\},$$

where $M<\min\{1,L_1,L_2\}$,

$$L_1=\Big(\frac{\int_1^e\Big([a(\cos\lambda(e-s)+1)]^{\sigma}M^{-\sigma}\varphi(s,1,1)+(Mb)^{-\sigma}(\ln s)^{-2\sigma}\psi(s,1,1)\Big)ds}{\lambda\sin\lambda(e-1)}\Big)^{-1},$$

$$L_2=\int_1^e M^{\sigma}[a(\cos\lambda(e-s)+1)]^{-\sigma}G_2(s,s)\psi(s,1,1)ds,$$

in which

$$a=\max\{\frac{N_2}{\Gamma(\alpha)},1\},b=\min\{\frac{N_1}{\Gamma(\alpha)},1\},$$

and

$$N_1=\int_1^e\ln\tau(1-\ln\tau)^{\alpha-2}\cos\lambda(e-\tau)\frac{d\tau}{\tau},\ N_2=\int_1^e\cos\lambda(e-\tau)\frac{d\tau}{\tau},$$

then $\tilde{P_0}$ is a normal and solid cone of the Banach space E.

To establish the uniqueness of positive solution to problem (1.1), we define an operator $T:\tilde{P_0}\times\tilde{P_0}\to\tilde{P_0}$ by

$$T(u,v)(t)=\int_1^e G_2(t,s)\Big[\varphi(s,\int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}+\frac{1}{n},u(s)+\frac{1}{n})+\psi(s,\int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}+\frac{1}{n},v(s)+\frac{1}{n})\Big]ds.$$

Firstly, we prove that $T:\tilde{P_0}\times\tilde{P_0}\to\tilde{P_0}$. In fact, for $u\in\tilde{P_0}$, similar to (3.1), we have

$$\frac{MN_1(\ln s)^{\alpha-1}}{\Gamma(\alpha)}\leq\int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}\leq\frac{N_2(\ln s)^{\alpha-1}}{M\Gamma(\alpha)}\leq\frac{N_2}{M\Gamma(\alpha)}. \tag{3.4}$$

Let $f(s)=\ln s-\cos\lambda(e-s),s\in[1,e]$ and then

$$f'(s)=\frac{1}{s}-\lambda\sin\lambda(e-s)=\frac{1}{s}g(s),$$

which $g(s)=1-\lambda s\sin\lambda(e-s)$. Since $0<\lambda\leq\frac{1}{\sqrt{e(e-1)}}$, we have $g(s)\geq 0$, which means that $f'(s)\geq 0$. Therefore, $f(s)$ is increasing on $[1,e]$, which means that $f(s)\leq f(e)=0$. This implies that $\ln s\leq\cos\lambda(e-s)$, i.e., $(\ln s)^2\leq\cos^2\lambda(e-s)$ for $s\in[1,e]$.

From (H3)-(H4) and Remark 3.4, for any $u, v \in \tilde{P}_0$, we have

$$\begin{aligned}
\varphi(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau} + \frac{1}{n}, u(s) + \frac{1}{n}) &\leq \varphi(s, \frac{N_2(\ln s)^{\alpha-1}}{M\Gamma(\alpha)} + 1, \frac{1}{M}\cos\lambda(e-s) + 1) \\
&\leq M^{-\sigma}\varphi(s, \frac{N_2(\ln s)^{\alpha-1}}{\Gamma(\alpha)} + 1, \cos\lambda(e-s) + 1) \\
&\leq a^{\sigma}M^{-\sigma}\varphi(s, (\ln s)^{\alpha-1} + 1, \cos\lambda(e-s) + 1) \\
&\leq a^{\sigma}M^{-\sigma}\varphi(s, \ln s + 1, \cos\lambda(e-s) + 1) \\
&\leq [a(\cos\lambda(e-s) + 1)]^{\sigma}M^{-\sigma}\varphi(s, 1, 1),
\end{aligned}$$

and

$$\begin{aligned}
\psi(s, \int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau} + \frac{1}{n}, v(s) + \frac{1}{n}) &\leq \psi(s, \frac{MN_1(\ln s)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{n}, M\cos\lambda(e-s) + \frac{1}{n}) \\
&\leq \psi(s, \frac{MN_1(\ln s)^{\alpha-1}}{\Gamma(\alpha)}, M\cos\lambda(e-s)) \\
&\leq M^{-\sigma}\psi(s, \frac{N_1(\ln s)^{\alpha-1}}{\Gamma(\alpha)}, \cos\lambda(e-s)) \\
&\leq (Mb)^{-\sigma}\psi(s, (\ln s)^{\alpha-1}, \cos\lambda(e-s)) \\
&\leq (Mb)^{-\sigma}\psi(s, (\ln s)^{2}, \cos^2\lambda(e-s)) \\
&\leq (Mb)^{-\sigma}(\ln s)^{-2\sigma}\psi(s, 1, 1).
\end{aligned}$$

Consequently, Lemma 2.4 yield

$$\begin{aligned}
T(u,v) &= \int_1^e G_2(t,s)\Big[\varphi(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau} + \frac{1}{n}, u(s) + \frac{1}{n}) + \psi(s, \int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau} + \frac{1}{n}, v(s) + \frac{1}{n})\Big]ds \\
&\leq \frac{\cos\lambda(e-t)}{\lambda\sin\lambda(e-1)}\int_1^e \Big([a(\cos\lambda(e-s)+1)]^{\sigma}M^{-\sigma}\varphi(s,1,1) + (Mb)^{-\sigma}(\ln s)^{-2\sigma}\psi(s,1,1)\Big)ds \\
&\leq \frac{1}{M}\cos\lambda(e-t).
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\psi(s, \int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau} + \frac{1}{n}, v(s) + \frac{1}{n}) &\geq \psi(s, \frac{N_2(\ln s)^{\alpha-1}}{M\Gamma(\alpha)} + \frac{1}{n}, \frac{1}{M}\cos\lambda(e-s) + \frac{1}{n}) \\
&\geq M^{\sigma}\psi(s, \frac{N_2(\ln s)^{\alpha-1}}{\Gamma(\alpha)} + 1, \cos\lambda(e-s) + 1) \\
&\geq M^{\sigma}a^{-\sigma}\psi(s, (\ln s)^{\alpha-1} + 1, \cos\lambda(e-s) + 1) \\
&\geq M^{\sigma}a^{-\sigma}\psi(s, \ln s + 1, \cos\lambda(e-s) + 1) \\
&\geq M^{\sigma}[a(\cos\lambda(e-s) + 1)]^{-\sigma}\psi(s, 1, 1).
\end{aligned}$$

Applying Lemma 2.4, we also have

$$\begin{aligned}
T(u,v) &= \int_1^e G_2(t,s)\Big[\varphi(s, \int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}, u(s)) + \psi(s, \int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}, v(s))\Big]ds \\
&\geq \int_1^e G_2(t,s)\psi(s, \int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}, v(s))ds \\
&\geq \cos\lambda(e-t)\int_1^e M^{\sigma}[a(\cos\lambda(e-s)+1)]^{-\sigma}G_2(s,s)\psi(s,1,1)ds \\
&\geq M\cos\lambda(e-t).
\end{aligned}$$

Hence, $T:\tilde{P_0}\times\tilde{P_0}\to\tilde{P_0}$.

Next, we shall show that $T:\tilde{P_0}\times\tilde{P_0}\to\tilde{P_0}$ is a mixed monotone operator. In fact, for any $u_1,u_2\in\tilde{P_0}$ with $u_1\le u_2$, from the monotonicity of $\int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}$ and φ, we have

$$\begin{aligned}
&T(u_1,v)\\
=\ &\int_1^e G_2(t,s)\Big[\varphi(s,\int_1^e G_1(s,\tau)u_1(\tau)\frac{d\tau}{\tau}+\frac{1}{n},u_1(s)+\frac{1}{n})+\psi(s,\int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}+\frac{1}{n},v(s)+\frac{1}{n})\Big]ds\\
\le\ &\int_1^e G_2(t,s)\Big[\varphi(s,\int_1^e G_1(s,\tau)u_2(\tau)\frac{d\tau}{\tau}+\frac{1}{n},u_2(s)+\frac{1}{n})+\psi(s,\int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}+\frac{1}{n},v(s)+\frac{1}{n})\Big]ds\\
=\ &T(u_2,v),
\end{aligned}$$

which implies that

$$T(u_1,v)\le T(u_2,v),\ \ for\ \ u_1\le u_2, \tag{3.5}$$

that is, $T(u,v)$ is non-decreasing in u for any $v\in\tilde{P_0}$. Similar to (3.5), if $v_1,v_2\in\tilde{P_0}$ with $v_1\ge v_2$, the following formula is also valid

$$T(u,v_1)(t)\ge T(u,v_2)(t),u\in\tilde{P_0}.$$

Hence, $T:\tilde{P_0}\times\tilde{P_0}\to\tilde{P_0}$ is a mixed monotone operator.

Finally, we prove that the operator T satisfies the condition (3.3). In fact, for any u, $v\in\tilde{P_0}$ and $0<c<1$, it follows from (H4) that

$$\begin{aligned}
T(cu,c^{-1}v)(t)\ &=\ \int_1^e G_2(t,s)\Big[\varphi(s,c\int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}+\frac{1}{n},cu(s)+\frac{1}{n})\\
&\quad+\psi(s,c^{-1}\int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}+\frac{1}{n},c^{-1}v(s)+\frac{1}{n})\Big]ds\\
&\ge\ \int_1^e G_2(t,s)c^{\sigma}\Big[\varphi(s,\int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}+\frac{1}{n},u(s)+\frac{1}{n})\\
&\quad+\psi(s,\int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}+\frac{1}{n},v(s)+\frac{1}{n})\Big]ds\\
&=\ c^{\sigma}\int_1^e G_2(t,s)\Big[\varphi(s,\int_1^e G_1(s,\tau)u(\tau)\frac{d\tau}{\tau}+\frac{1}{n},u(s)+\frac{1}{n})\\
&\quad+\psi(s,\int_1^e G_1(s,\tau)v(\tau)\frac{d\tau}{\tau}+\frac{1}{n},v(s)+\frac{1}{n})\Big]ds\\
&=\ c^{\sigma}T(u,v)(t),t\in[1,e].
\end{aligned}$$

From Lemma 3.6, we can see that the Hadamard-type fractional differential equation (2.1) has a unique positive solution u^* such that $T(u^*,u^*)=u^*$. Consequently, problem (1.1) has a unique positive solution $x^*\in\tilde{P_0}$, i.e. $x^*(t)=\int_1^e G_1(t,s)u(s)\frac{ds}{s}$ is the unique positive solution of BVP (1.1), and

$$M\cos\lambda(e-t)\le x^*(t)\le\frac{1}{M}\cos\lambda(e-t).$$

Moreover, for any initial $u_0,v_0\in\tilde{P_0}$, we can construct successively two sequences $\{u_n\}$ and $\{v_n\}$ by

$$u_n=T(u_{n-1},v_{n-1}),\qquad v_n=T(v_{n-1},u_{n-1}),n=1,2,...,$$

and the iterative sequence $u_n(t)$, $v_n(t)$ converges uniformly to u^* on $[1,e]$ as $n\to\infty$, i.e.,

$$u_n\to-\mathcal{D}^{\alpha}_{1+}x^*(t),v_n\to-\mathcal{D}^{\alpha}_{1+}x^*(t)\ \ as\ \ n\to\infty.$$

Therefore, the Theorem 3.3 is proved. $\diamond$

4. **Stability**

In this section, we consider the Ulam-Hyers-Rassias stability of the solution of equation (1.1). We first give the definition of stability:

Definition 4.1. Equation (1.1) is said to be Ulam-Hyers-Rassias stable with respect to $\Psi \in E$ if there exists a nonzero positive real number Λ such that, for every $\varepsilon > 0$ and each solution $y \in E$ of the inequality

$$|(\mathcal{D}_{1^+}^{\alpha} y(t))'' + \lambda^2 \mathcal{D}_{1^+}^{\alpha} y(t) + f(t, y(t), -\mathcal{D}_{1^+}^{\alpha} y(t))| \leq \varepsilon \Psi(t),\ t \in [1, e], \tag{4.1}$$

there exists a solution $x(t)$ of problem (1.1) such that

$$|x(t) - y(t)| \leq \Lambda_{\varepsilon} \Psi(t),\ t \in [0, 1].$$

Theorem 4.2. Assume that $f : [1, e] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}^+$ is continuous and (H1) holds, then the problem (1.1) is Ulam-Hyers-Rassias stable if $(\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2) \frac{e-1}{\lambda \sin \lambda(e-1)} < 1$.

Proof. Let $y \in C[1, e]$ be the solution of the inequality (4.1), then

$$|(\mathcal{D}_{1^+}^{\alpha} y(t))'' + \lambda^2 \mathcal{D}_{1^+}^{\alpha} y(t) + f(t, y(t), -\mathcal{D}_{1^+}^{\alpha} y(t))| \leq \varepsilon \Psi(t),\ t \in [1, e].$$

Let $-\mathcal{D}_{1^+}^{\alpha} y(t) = v(t)$ and then similar to the discussion of x, we get that, for $\varepsilon > 0$,

$$|v(t) - \int_1^e G_2(t, s) f(s, \int_1^e G_1(s, \tau) v(\tau) \frac{d\tau}{\tau}, v(s)) ds| \leq \varepsilon \Psi(t),\ t \in [1, e].$$

Hence, the stability of the solution of equation (1.1) can be equivalent to the stability of equation (2.1).

According to the discussion of Theorem 3.1, the problem (2.1) has a solution $u(t)$ satisfying

$$u(t) = \int_1^e G_2(t, s) f(s, \int_1^e G_1(s, \tau) u(\tau) \frac{d\tau}{\tau}, u(s)) ds.$$

Then for $t \in [1, e]$, we have from Lemma 2.4 that

$$\begin{aligned}
|u(t) - v(t)| &= |v(t) - \int_1^e G_2(t, s) f(s, \int_1^e G_1(s, \tau) u(\tau) \frac{d\tau}{\tau}, u(s)) ds| \\
&\leq |v(t) - \int_1^e G_2(t, s) f(s, \int_1^e G_1(s, \tau) v(\tau) \frac{d\tau}{\tau}, v(s)) ds| \\
&\quad + |\int_1^e G_2(t, s) \Big(f(s, \int_1^e G_1(s, \tau) u(\tau) \frac{d\tau}{\tau}, u(s)) - f(s, \int_1^e G_1(s, \tau) v(\tau) \frac{d\tau}{\tau}, v(s)) \Big) ds| \\
&\leq \varepsilon \Psi(t) + \int_1^e G_2(t, s) (\mathcal{L}_1 \int_1^e G_1(s, \tau) (u(\tau) - v(\tau)) \frac{d\tau}{\tau} + \mathcal{L}_1 |u(s) - v(s)|) ds \\
&\leq \varepsilon \Psi(t) + (\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2) \|u - v\| \int_1^e G_2(t, s) ds \\
&\leq \varepsilon \Psi(t) + (\frac{\mathcal{L}_1}{\Gamma(\alpha)} + \mathcal{L}_2) \frac{e-1}{\lambda \sin \lambda(e-1)} \|u - v\|,
\end{aligned}$$

which yields

$$|v(t)-u(t)| \le \frac{\varepsilon}{1-(\frac{\mathcal{L}_1}{\Gamma(\alpha)}+\mathcal{L}_2)\frac{e-1}{\lambda\sin\lambda(e-1)}}\Psi(t)=\Lambda_\varepsilon\Psi(t),\ t\in[1,e].$$

Therefore, the problem (2.1) is Ulam-Hyers-Rassias stable, which implies that the solution of equation (1.1) is Ulam-Hyers-Rassias stable.$\diamond$

5. **Examples**

In this section, we will give two examples to illustrate the existence and uniqueness of the solution of the fractional equation.

Example 5.1. Considering the following equation:

$$\begin{cases} (-\mathcal{D}_{1^+}^{\frac{5}{2}}x(t))''-\frac{\pi^2}{4e^2}\mathcal{D}_{1^+}^{\frac{5}{2}}x(t)=t(x+|-\mathcal{D}^{\frac{5}{2}}x(t)|)^{\frac{1}{2}},\ t\in[1,e],\\ \\ x(1)=x'(1)=x'(e)=0, \mathcal{D}_{1^+}^{\frac{7}{2}}x(1)=\mathcal{D}_{1^+}^{\frac{7}{2}}x(e)=0.\end{cases}$$

Let $f(t,x(t),-\mathcal{D}^{\alpha}x(t))=t(x+|-\mathcal{D}^{\frac{5}{2}}x(t)|)^{\frac{1}{2}}$ and $\gamma=\frac{1}{2}$. Taking $h_1(t)\equiv 1$, $h_2(t)\equiv e$, then

$$(u+v)^{\frac{1}{2}}\le f(t,u,v)\le e(u+v)^{\frac{1}{2}},$$

which means (H2) is hold. Let $\alpha=\frac{5}{2}$ and $\lambda=\frac{\pi}{2e}$, then

$$G_1(t,s)=\frac{1}{\Gamma(\frac{5}{2})}\begin{cases}(\ln t)^{\frac{3}{2}}(1-\ln s)^{\frac{1}{2}}-(\ln t-\ln s)^{\frac{3}{2}}, & 1\le s\le t\le e,\\ (\ln t)^{\frac{3}{2}}(1-\ln s)^{\frac{1}{2}}, & 1\le t\le s\le e,\end{cases}$$

and

$$G_2(t,s)=\frac{2e}{\pi\sin\frac{\pi}{2e}(e-1)}\begin{cases}\cos\frac{\pi}{2e}(1-t)\cos\frac{\pi}{2e}(s-e)+\sin\frac{\pi}{2e}(e-1)\sin\frac{\pi}{2e}(t-s), & 1\le s\le t\le e,\\ \cos\frac{\pi}{2e}(1-t)\cos\frac{\pi}{2e}(s-e), & 1\le t\le s\le e.\end{cases}$$

In addition,

$$M_1=\frac{2e}{\pi\sin\frac{\pi}{2e}(e-1)}\int_1^e\cos\frac{\pi}{2e}(1-s)\cos\frac{\pi}{2e}(s-e)ds\approx 2.46679963$$

and

$$M_2=\Big(1+\frac{1}{\Gamma(\frac{5}{2})}\Big)^{\frac{1}{2}}e(e-1)(\frac{\pi}{2e}\sin\frac{\pi}{2e}(e-1))^{-1}\approx 10.5179295.$$

Let $L=38000$, then $M_1L^{\frac{1}{2}}>1$ and $M_2L^{-\frac{1}{2}}>1$ is hold. Therefore, this example satisfies the conditions of Theorem 3.2, so the equation has at least one positive solution.

Example 5.2. Considering the following Hadamard-type fractional differential equation:

$$\begin{cases}(-\mathcal{D}_{1^+}^{\frac{5}{2}}x(t))''-\frac{\pi^2}{4e^2}\mathcal{D}_{1^+}^{\frac{5}{2}}x(t)=t[x^{\frac{1}{5}}(t)-(\mathcal{D}_{1^+}^{\frac{5}{2}}x(t))^{\frac{1}{4}}]+t^{-\frac{1}{2}}[x^{-\frac{1}{8}}(t)-(\mathcal{D}_{1^+}^{\frac{5}{2}}x(t))^{-\frac{1}{6}}],\ t\in[1,e],\\ \\ x(1)=x'(1)=x'(e)=0, \mathcal{D}_{1^+}^{\frac{7}{2}}x(1)=\mathcal{D}_{1^+}^{\frac{7}{2}}x(e)=0.\end{cases}$$

Let $\varphi(t,u,v)=t(u^{\frac{1}{5}}+v^{\frac{1}{4}})$, $\psi(t,u,v)=t^{-\frac{1}{2}}(u^{-\frac{1}{8}}+v^{-\frac{1}{6}})$ and $\sigma=\frac{1}{4}$, then

$$\varphi(t,cu,cv)=t(c^{\frac{1}{5}}u^{\frac{1}{5}}+c^{\frac{1}{4}}v^{\frac{1}{4}})\ge c^{\frac{1}{4}}t(u^{\frac{1}{5}}+v^{\frac{1}{4}})=c^{\frac{1}{4}}\varphi(t,u,v)$$

and

$$\psi(t,c^{-1}u,c^{-1}v)=t^{-\frac{1}{2}}(c^{\frac{1}{8}}u^{-\frac{1}{8}}+c^{\frac{1}{6}}v^{-\frac{1}{6}})\geq c^{\frac{1}{4}}t^{-\frac{1}{2}}(u^{-\frac{1}{8}}+v^{-\frac{1}{6}})=c^{\frac{1}{4}}\psi(t,u,v),$$

for any $u,v>0$ and $0<c<1$. Clearly, $\psi:[1,e]\times(0,+\infty)^2\to[0,+\infty)$ and $\varphi:[1,e]\times[0,+\infty)^2\to[0,+\infty)$ are continuous. Moreover, $\varphi(t,u,v)$ is non-decreasing and $\psi(t,u,v)$ is nonincreasing in $u,v>0$ respectively. Thus by Theorem 3.3, the fractional differential equation has a unique positive solution x^*, and there exists a constant $0<M<1$ such that

$$M\cos\frac{\pi}{2e}(e-t)\leq x^*(t)\leq\frac{1}{M}\cos\frac{\pi}{2e}(e-t).$$

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