Radial solutions for Dirichlet systems with Monge-Ampère operator

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Abstract. We are concerned with a Dirichlet system, involving the Monge-Ampère operator $\det D^2u$ in a ball in $\mathbb{R}^N$. Based on the Leray-Schauder degree, we first obtain the existence of radial solutions for a class of differential systems with general nonlinearities. In addition, we prove that such a system admits positive solutions when nonlinearities satisfy sub- or superlinear growth near origin. Finally, by using the lower and upper solution method, and constructing the subsolution and supersolution, we show the existence and multiplicity of nontrivial radial solutions for Dirichlet systems with Monge-Ampère operator and Lane-Emden type nonlinearities with two parameters.

Mathematics Subject Classification (2010). 34B15, 34B18, 35J47, 35J96.

Keywords. Monge-Ampère operator; Radial solutions; Existence; Leray-Schauder degree; Lower and upper solutions method; Lane-Emden nonlinearities.

1. Introduction

In this paper, we deal with the Dirichlet system involving the Monge-Ampère operator

$$\begin{cases} \det D^2u = f_1(|x|, -u, -v) & \text{in } \mathcal{B}(R), \\ \det D^2v = f_2(|x|, -u, -v) & \text{in } \mathcal{B}(R), \\ u = v = 0 & \text{on } \partial\mathcal{B}(R), \end{cases} \tag{1.1}$$

where $\det D^2u$ is the determinant of the Hessian matrix $\frac{\partial^2 u}{\partial x_i \partial x_j}$ of u, $\mathcal{B}(R) = \{x \in \mathbb{R}^N : |x| < R\}$, $N \geq 2$ is an integer and the functions f_1 and $f_2 : [0, R] \times [0, \infty)^2 \to [0, \infty)$ are continuous.

We know that the Monge-Ampère equation is a fully nonlinear, degenerate elliptic equation, which arising in several problems in the areas of analysis and geometry, such as prescribed Gaussian curvature problem, affine geometry, and optimal transportation problem, see [3, 8, 17, 21], and the references therein. In recent years, a particular attention was paid to Dirichlet problems (for a single equation) involving the Monge-Ampère operator, either in a general bounded domain [7, 16, 27] or in a ball [18, 30, 22]. With the application of operator theory in analysis and geometry, systems with classical Laplacian operators or other more general elliptic operators have introduced some new concrete phenomena to the discussion that are more complex and difficult than the study of single equations. The establishment of this view can be found in references [7, 18, 30, 24, 29] and the references therein.

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Monge-Ampère equations figure in various geometric problems [10, 11]. As we know, the classical model of the Monge-Ampère problem is

$$\begin{cases} \det D^2u = \lambda f(-u) & \text{in } B, \\ u = 0 & \text{on } \partial B, \end{cases} \tag{1.2}$$

where $B = \{x \in \mathbb{R}^n : |x| < 1\}$ is the unit ball in $\mathbb{R}^n$, λ is a nonnegative parameter and $f : \mathbb{R} \to \mathbb{R}$ is a continuous function. In 1984, L. Caffarelli et al. [3] took up an extension of the problem (1.2) to more general domain, always in a bounded domain Ω with strictly convex C^∞ boundary $\partial\Omega$. After that, Kutev [14] studied the existence and uniqueness of strictly convex solutions of problem (1.2) dependent on parameter p with the nonnegative parameter $\lambda = 1$ and $f(-u) = (-u)^p$. Subsequently, based on his work, Hu and Wang [13] studied the problem (1.2) and established several criteria for the existence of strictly convex solutions with or without an eigenvalue parameter. More recently, under the case $\det D^2u = (\lambda f(-u))^N$, Luo et al. [16] established some interesting results, by using the bifurcation technique.

As for the study of system with Monge-Ampère operator, we know that the most common are coupled system and n-dimensional system, so the transformation from a single equation to a system equation brings us many difficulties in the process of discussing problems, such as operator coupling problem and space transformation problem. In recent years, we have noticed that many scholars focused on kinds of systems, see for instance, [4, 7, 9, 10, 11, 24, 29] and the references therein. However, to our best knowledge, it is a few results on such systems with Monge-Ampère operator, see for instance, [7, 18, 24, 30] and the references therein. Among them, the typical one is [24], in this work,Wang considered the system problem with Monge-Ampère operators

$$\begin{cases} \det D^2u = f_1(-v) & \text{in } B, \\ \det D^2v = f_2(-u) & \text{in } B, \\ u = v = 0 & \text{on } \partial B \end{cases} \tag{1.3}$$

and obtained the existence of radial convex solutions in sub- and superlinear cases, where $B = \{x \in \mathbb{R}^N : |x| < 1\}$ and $f_i (i = 1, 2)$ are continuous. In constract, Liu et al. [18] introduced some new growth conditions on the nonlinearities f_1 and f_2, several new existence and multiplicity results of radial solutions for (1.3) are obtained. Specially, for the case $f_1(-v) = (-v)^\alpha$, $f_2(-u) = (-u)^\beta$, Zhang et al. [30] obtained the existence, uniqueness and nonexistence of radial convex solutions under some corresponding assumptions on α, β. When $\alpha > 0$, $\beta > 0$ and $\alpha\beta = N^2$, they also considered a corresponding eigenvalue problem in a more general smooth, bounded and strictly convex domain.

Inspired by the above literatures, in this paper, we first consider a more general system (1.1) in section 2, and obtain the existence of nontrivial radial solutions of problem (1.1) when the nonlinearities satisfy or does not satisfy the quasi-monotone nondecreasing condition. Then, in section 3, we consider the existence of the radial solution of problem (1.1) when the nonlinearities satisfy sub- or suplinear conditions, respectively. In addition, we discuss the monotone property of the radial solution.

At the same time, it is worth noting that Lane-Emden type plays an important role in various kinds of nonlinearities, and its specific form is $k_1u^p + k_2v^q$, see [4, 26] or $k_3u^\alpha v^\beta$, see [12, 15]. In recent decades, many scholars have studied the existence of solutions for various problems with Lane-Emden type nonlinear terms, such as the Laplace problem and the mean curvature problem in Minkowski apsce, see literatures [10, 11, 5, 6, 25] for details. However, to our best knowledge, it is a few results on such problems with Monge-Ampère operator. It appears as a natural direction the study of systems involving the Monge-Ampère operator

$$\begin{cases} \det D^2u = \lambda_1\nu_1(|x|)(-u)^{p_1}(-v)^{q_1} & \text{in } \mathcal{B}(R), \\ \det D^2v = \lambda_2\nu_2(|x|)(-u)^{p_2}(-v)^{q_2} & \text{in } \mathcal{B}(R), \\ u = v = 0 & \text{on } \partial\mathcal{B}(R), \end{cases} \tag{1.4}$$

where weight functions $\nu_1, \nu_2 \in C([0,R],[0,\infty))$ with $\nu_1(r) > 0 < \nu_2(r)$ for all $r \in (0,R]$, p_1, q_2 are nonnegative and q_1, p_2 are positive exponents. We will show that there exist $\lambda_1^* > 0 < \lambda_2^*$ such that system (1.4) admits a radial solution (u,v) for all $\lambda_1 > \lambda_1^*$, $\lambda_2 > \lambda_2^*$, and both u and v are decreasing. Further, by applying the existence of a lower solution in a cone of positive functions and the estimation of Leray-Schauder degree, we also prove that if $\min\{p_1, q_2\} > N$, then there exists a continuous curve Γ that divides the first quadrant into two disjoint unbounded open sets, $\mathcal{O}_1$ and $\mathcal{O}_2$ such that the system (1.4) has only the trivial solution, at least one or two radial solutions according to $(\lambda_1,\lambda_2) \in \mathcal{O}_1$, $(\lambda_1,\lambda_2) \in \Gamma$ or $(\lambda_1,\lambda_2) \in \mathcal{O}_2$, respectively. In this discussion, the set $\mathcal{O}_1$ is adjacent to the coordinate axes $0\lambda_1$ and $0\lambda_2$, and the curve Γ is asymptotically close to two straight lines parallel to the coordinate axes $0\lambda_1$, $0\lambda_2$.

The rest of the paper is organized as follows. In section 2, we present some preparations and the existence of nontrivial solutions for problem (1.1). Section 3 is devoted to the cases when f_1 and f_2 have a sub- or suplinear behavior near origin. In both cases, we obtain the existence of the radial solutions of problem (1.1). The lower and upper solution method and some degree estimations in the superlinear case are presented in Section 4 and our main goal of this work is the non-existence, existence and multiplicity results for system (1.4) with Lane-Emden type nonlinearities are stated and proved in Section 5.

2. Preliminaries

Throughout this paper, we denote $C := C[0,R]$ with the usual sup-norm $\|\cdot\|_\infty$. The space $C^1 = C^1[0,R]$ will be equipped with the norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$, and the product space $C^1 \times C^1$ will be endowed with the norm $\|(u,v)\| = \max\{\|u\|_\infty, \|v\|_\infty\} + \max\{\|u'\|_\infty, \|v'\|_\infty\}$. We consider the closed subspace

$$P := \{(u,v) \in C^1 \times C^1 : u'(0) = u(R) = 0 = v(R) = v'(0)\}$$

and its closed, convex cone

$$K := \{(u,v) \in P : u \geq 0 \leq v \quad \text{on} \quad [0,R]\}$$

be a convex cone, $\widetilde{B}(\rho) := \{(u,v) \in P : \|(u,v)\| < \rho\}$, $B(\rho) := K \cap \widetilde{B}(\rho)$.

Let us seek radial solutions of (1.1). As usual, for radial solution $u(r)$ with $r = \sqrt{\Sigma_{i=1}^N x_i^2}$, the Monge-Ampère operator simply becomes

$$\det D^2 u = \frac{(u')^{N-1} u''}{r^{N-1}} = \frac{\big((u')^N\big)'}{N r^{N-1}}. \tag{2.1}$$

For convenience, let $u = -u(r)$ and $v = -v(r)$, then the Dirichlet problem (1.1) can convert to the following boundary value problem

$$\begin{cases} [(-u'(r))^N]' = N r^{N-1} f_1(r,u,v), & r \in (0,R), \\ [(-v'(r))^N]' = N r^{N-1} f_2(r,u,v), & r \in (0,R), \\ u(r) > 0, \quad v(r) > 0, & r \in (0,R), \\ u'(0) = u(R) = v(R) = v'(0) = 0. \end{cases} \tag{2.2}$$

Denote a couple of nonnegative functions $(u,v) \in C^1[0,R] \times C^1[0,R]$ as a solution of (2.2) with $r \mapsto (u'(r))^N$ and $r \mapsto (v'(r))^N$ of class C^1 on $[0,R]$. We say that $u \in C$ is positive if $u > 0$ on $[0,R)$. By a positive solution (u,v) of (2.2), we mean (u,v) satisfies (2.2) and both u and v are positive.

Let us define linear operators

$$S : C \to C, \quad Su(r) = \Big(\int_0^r N t^{N-1} u(t)dt\Big)^{\frac{1}{N}}, \quad r \in (0,R];$$

$$T : C \to C^1, \quad Tu(r) = \int_r^R u(t)dt, \quad r \in [0,R].$$

It is easy to see that S is compact and T is bounded. Hence, the nonlinear operator $T \circ S : C \to C^1$ is compact. Denoting by $N_{f_i} : C \times C \to C$ the Nemytskii operator associated to $f_i (i = 1, 2)$ as follows

$$N_{f_i} = f_i(\cdot, u(\cdot), v(\cdot)), \quad (u, v) \in C,$$

then N_{f_i} $(i = 1, 2)$ are continuous and map a bounded set to a bounded set. A couple of functions (u, v) is a solution of (2.2) if and only if it is a fixed point of the compact nonlinear operator

$$\mathcal{N}_f : K \to K, \quad \mathcal{N}_f = (T \circ S \circ N_{f_1}, T \circ S \circ N_{f_2}).$$

In the following statement, we denote the Leray-Schauder degree by d_{LS}.

Based on this, we have the following results:

Lemma 2.1 *All fixed points $(u, v) \in \overline{B}(\rho)$ of $\mathcal{N}_f$ satisfy*

$$\|\mathcal{N}_f(u, v)\| < MR(R + 1). \tag{2.3}$$

Proof Let $M^N := \max_{[0,R]\times\overline{\mathcal{B}}(\rho)}\{f_1(\tau, u, v), f_2(\tau, u, v)\}$, where $\mathcal{B}(\rho)$ is a circle with radius ρ and with its center at the origin. From the above definitions of operators, we know that

$$T \circ S \circ N_{f_i}(u, v) = \int_r^R \Big(\int_0^t N\tau^{N-1} f_i(\tau, u(\tau), v(\tau))d\tau \Big)^{\frac{1}{N}} dt, \quad r \in [0, R].$$

Since f is bounded in $[0, R] \times \overline{B}\rho$, it follows that

$$\|u'\|_\infty \le MR, \quad \|v'\|_\infty \le MR, \quad \|u\|_\infty < MR^2, \quad \|v\|_\infty < MR^2,$$

Therefore (2.3) holds from the above inequalities.

Lemma 2.2 *For all $d \ge MR(R + 1)$,*

$$d_{LS}(I - \mathcal{N}_f, B(d), 0) = 1.$$

Proof We consider the compact homotopy $\mathcal{H} : [0, 1] \times \overline{B}(d) \to P$ given by $\mathcal{H}(t, \cdot) = t\mathcal{N}_f(\cdot)$. Since

$$\mathcal{H}(t, (u, v)) = t\mathcal{N}_f(u, v) \le t\|\mathcal{N}_f(u, v)\| < MR(R + 1), \qquad (t, (u, v)) \in [0, 1] \times \overline{B}(d)$$

that is $\mathcal{H}(t, (u, v)) \subset B(MR(R + 1))$, which derives that $\mathcal{H}(t, \cdot)$ has no fixed point in $\partial B(d)$, $t \in [0, 1]$. Then, the following formula can be obtained from the homotopy invariance of the Leray-Schauder degree

$$d_{LS}(I - \mathcal{N}_f, B(d), 0) = d_{LS}(I, B(d), 0) = 1 \quad \text{for all} \quad d \ge MR(R + 1).$$

□

Now, we choose the constants

$$b \in (0, R), \quad 0 < \alpha < R - b, \quad d \ge MR(R + 1) + \alpha \tag{2.4}$$

and introduce a continuous function $\phi : P \to \mathbb{R}$ defined as follows

$$\phi(u, v) = \min \Big\{ \min_{[0,b]} u(t), \min_{[0,b]} v(t) \Big\}$$

and we put

$$D_\alpha := \{(u, v) \in P : \phi(u, v) < \alpha\},$$
$$U_\alpha := \{(u, v) \in B(d) : \phi(u, v) < \alpha\}.$$

Notice that D_α is an open set in P, hence $U_\alpha = K \cap \widetilde{B}(d) \cap D_\alpha$ is a bounded open nonempty subset of K, the non-nullity of U_α is guaranteed by $(0, 0) \in U_\alpha$.

Lemma 2.3. *If $(u, v) \in \partial U_\alpha$, then $\|(u, v)\| = d$ or $\phi(u, v) = \alpha$.*

Proof Since $\partial U_\alpha \subset \big(K \cap \partial[\widetilde{B}(d) \cap D_\alpha]\big) \subset \partial[\widetilde{B}(d) \cap D_\alpha]$, it derives that $(u, v) \in \overline{\widetilde{B}(d) \cap D_\alpha} \setminus [\widetilde{B}(d) \cap D_\alpha]$. From $(u, v) \in \overline{\widetilde{B}(d) \cap D_\alpha}$, we get that

$$\|(u, v)\| \le d \quad \text{and} \quad \phi(u, v) \le \alpha,$$

while from $(u, v) \notin \widetilde{B}(d) \cap D_\alpha$, we have

$$\|(u, v)\| \ge d \quad \text{or} \quad \phi(u, v) \ge \alpha,$$

it leads to $\|(u,v)\| = d$ or $\phi(u,v) = \alpha$. $\square$

Lemma 2.4. *If*

$$\mathcal{N}_f(u,v) + t(\alpha,\alpha) \neq (u,v), \text{ for all } t \in [0,1] \text{ and } (u,v) \in K \text{ with } \phi(u,v) = \alpha, \tag{2.5}$$

then

$$d_{LS}(I - \mathcal{N}_f, U_\alpha, 0) = 0.$$

Proof At first, we consider the homotopy $\mathcal{H} : [0,1] \times \overline{U}_\alpha \to \overline{U}_\alpha$ given as follows

$$\mathcal{H}(t,(u,v)) = \mathcal{N}_f(u,v) + t(\alpha,\alpha).$$

We claim that $(0,0) \notin (I - \mathcal{H}(t,(u,v)))(\partial U_\alpha)$. Otherwise, there exist $t_0 \in [0,1]$ and $(u_0,v_0) \in \partial U_\alpha$ with $(u_0,v_0) = \mathcal{N}_f(u_0,v_0) + t_0(\alpha,\alpha)$. From Lemma 2.3, we know that $\|(u_0,v_0)\| = d$ or $\phi(u_0,v_0) = \alpha$. On account of (2.3), $\|(u_0,v_0)\| = d$ derives that

$$d = \|(u_0,v_0)\| = \|\mathcal{N}_f(u_0,v_0) + t_0(\alpha,\alpha)\| \leq \|\mathcal{N}_f(u_0,v_0)\| + t_0\alpha < MR(R+1) + \alpha,$$

which is a contradiction. In addition, we know that $\phi(u_0,v_0) = \alpha$ contradicts with (2.5).

On the contrary, we assume that $d_{LS}(I - \mathcal{N}_f, U_\alpha, 0) \neq 0$. Then, by the invariance under homotopy of the Leray-Schauder degree, we infer that

$$d_{LS}(I - \mathcal{H}(1), U_\alpha, 0) = d_{LS}(I - \mathcal{H}(0), U_\alpha, 0) = d_{LS}(I - \mathcal{N}_f, U_\alpha, 0) \neq 0.$$

Then, there exists $(u^*,v^*) \in U_\alpha$ such that

$$\mathcal{N}_f(u^*,v^*) + (\alpha,\alpha) = (u^*,v^*).$$

For $\mathcal{N}_f(u^*,v^*) \in K$, we get the contradiction

$$\alpha > \phi(u^*,v^*) = \phi(\mathcal{N}_f(u^*,v^*) + (\alpha,\alpha)) \geq \phi(\alpha,\alpha) = \alpha.$$

$\square$

Theorem 2.5. *If (2.5) is satisfied, then problem (2.2) has a nontrivial solution in $\overline{B}(d) \setminus U_\alpha$.*

Proof For any $(u,v) \in \overline{U}_\alpha$, $(u,v) \neq (0,0)$, Lemma 2.4 implies that $d_{LS}(I - \mathcal{N}_f, U_\alpha, 0) = 0$. In addition, Lemma 2.2 derives that $d_{LS}(I - \mathcal{N}_f, B(d), 0) = 1$.

Thus, there exists $(u,v)(\neq (0,0)) \in \overline{B}(d) \setminus U_\alpha$, we have

$$(I - \mathcal{N}_f)(u,v) = (0,0),$$

then (2.2) has a nontrivial solution. $\square$

Recall, a function $f = f(r,s,t) : [0,R] \times [0,\infty)^2 \to [0,\infty)$ is said to be *quasi-monotone nondecreasing* with respect to t (resp. s) if for fixed r, s (resp. r, t) one has

$$f(r,s,t_1) \leq f(r,s,t_2), \quad t_1 \leq t_2 \quad (\text{resp. } f(r,s_1,t) \leq f(r,s_1,t), \quad s_1 \leq s_2).$$

Theorem 2.6. *Assume that $f_i(r,s,t)$ $(i = 1,2)$ are quasi-monotone nondecreasing with respect to both s,t, together with*

$$\alpha < (R-b)\Big(\int_0^b N\tau^{N-1} f_i(\tau,\alpha,\alpha)d\tau\Big)^{\frac{1}{N}}. \tag{2.6}$$

Then, problem (2.2) has a nontrivial solution.

Proof We claim that (2.5) holds. Suppose on the contrary, there exist $t_0 \in [0,1]$ and $(u_0,v_0) \in K$ with $\phi(u_0,v_0) = \alpha$ such that $\mathcal{N}_f(u_0,v_0) + t_0(\alpha,\alpha) = (u_0,v_0)$.

Without loss of generality, we may assume that $\phi(u,v)=\min_{[0,b]}u(r)$. Then, for fixed $r_0\in[0,b]$, we get $\phi(u,v)=\alpha=u(r_0)$, $u(r)\geq u(r_0)$, $r\in[0,b]$ and

$$\begin{aligned}\alpha=u(r_0)&=\int_{r_0}^{R}\Big(\int_0^t N\tau^{N-1}f_1(\tau,u(\tau),v(\tau))d\tau\Big)^{\frac{1}{N}}dt+t\alpha\\&\geq\int_{b}^{R}\Big(\int_0^b N\tau^{N-1}f_1(\tau,u(\tau),v(\tau))d\tau\Big)^{\frac{1}{N}}dt\\&\geq\int_{b}^{R}\Big(\int_0^b N\tau^{N-1}f_1(\tau,u(r_0),v(\tau))d\tau\Big)^{\frac{1}{N}}dt\\&\geq\int_{b}^{R}\Big(\int_0^b N\tau^{N-1}f_1(\tau,\alpha,\alpha)d\tau\Big)^{\frac{1}{N}}dt\\&=(R-b)\Big(\int_0^b N\tau^{N-1}f_1(\tau,\alpha,\alpha)d\tau\Big)^{\frac{1}{N}},\end{aligned}$$

which contradicts (2.6). □

To get our main results, we shall need to introduce the following lemmas:

Lemma 2.7 *Assume that (u,v) is a nontrivial solution of problem (2.2) and*

(H_f^1) (i) $f_1(r,\xi,0)>0<f_2(r,0,\xi)$, $\quad\forall\xi>0,\quad r\in(0,R]$

holds, then $u\geq 0\leq v$ and either u or v is positive and strictly decreasing.

In addition, if $f_1(r,s,t)$ (resp. $f_2(r,s,t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and

(ii) $f_1(r,0,\xi)>0<f_2(r,\xi,0)$, $\quad\forall\xi>0,\quad r\in(0,R]$

holds, then (u,v) is a positive solution with both u and v strictly decreasing.

Proof Since

$$u'=-\Big(\int_0^t N\tau^{N-1}f_1(\tau,u,v)d\tau\Big)^{\frac{1}{N}}\leq 0,$$

which means that u is decreasing. Similarly, one obtains that v is decreasing. Then $u(R)=0$ implies that $u\geq 0$ and, analogously, $v\geq 0$. In this regard, if $u\equiv 0$, we have

$$v'=-\Big(\int_0^t N\tau^{N-1}f_2(\tau,0,v)d\tau\Big)^{\frac{1}{N}}.$$

It follows from $v(0)>0$ and (H_f^1) (i) that $v'<0$, therefore v is strictly decreasing and $v>0$ on $[0,R)$. Similarly, if $v\equiv 0$, which yields that u is strictly decreasing and $u>0$ on $[0,R)$.

Next, we prove that (u,v) is a positive solution with both u and v strictly decreasing when $f_1(r,s,t)$ (resp. $f_2(r,s,t)$) is quasi-monotone nondecreasing with respect to t (resp. s) and the condition (ii) holds. For this, we suppose that u is positive and we need to verify that v is also positive. If $v(0)=0$, then

$$v(0)=\int_0^R\Big(\int_0^t N\tau^{N-1}f_2(\tau,u(\tau),0)d\tau\Big)^{\frac{1}{N}}dt=0$$

follows that $f_2(r,u(r),0)=0$ for all $r\in[0,R]$, which contradicts with (H_f^1) (ii). Hence, $v(0)>0$.

Further, from the assumption that $f_2(r,s,t)$ is quasi-monotone nondecreasing with respect to s, we obtain

$$v'=-\Big(\int_0^t N\tau^{N-1}f_2(\tau,u(\tau),v(\tau))d\tau\Big)^{\frac{1}{N}}\leq-\Big(\int_0^t N\tau^{N-1}f_2(\tau,0,v(\tau))d\tau\Big)^{\frac{1}{N}}<0.$$

Hence, v is strictly decreasing. Similarly, u is strictly decreasing. □

Lemma 2.8 *Assume that*

(H_f^2) (i) $f_1(r,s,t)>0<f_2(r,s,t)$, $\quad\forall s,t>0,\quad r\in(0,R]$;

(ii) $f_1(r,\xi,0)=f_2(r,0,\xi)=0$, $\quad\forall\xi>0,\quad r\in(0,R]$.

If (u,v) is a nontrivial solution of problem (2.2), then $(u(r),v(r))$ $r\in[0,R)$ is a positive solution with both u and v are strictly decreasing.

Proof It follows from (H_f^2) (i) and

$$v' = -\Big(\int_0^t N\tau^{N-1} f_2(\tau, u, v)d\tau\Big)^{\frac{1}{N}} \tag{2.7}$$

that $v' < 0$, hence, v is strictly decreasing. Similarly, we have u is strictly decreasing.

In addition, $u(R) = 0$ yields that $u \geq 0$ (resp. $v(R) = 0$ yields that $v \geq 0$). We suppose that $u \equiv 0$, then from $v \not\equiv 0$, (H_f^2) (ii) and $v(R) = 0$ derives that $v \equiv 0$, which is a contradiction. Similarly, $v \not\equiv 0$. Thus (u, v) is a positive solution on $[0, R)$. □

Next, we make the hypothesis:

(H_f) The functions $f_i(r, s, t) : [0, R] \times [0, \infty)^2 \to [0, \infty)$ $(i = 1, 2)$ are continuous, quasi-monotone nondecreasing with respect to both s, t and satisfy

$$\int_0^b N\tau^{N-1} f_i(\tau, \alpha, \alpha)d\tau > 0, \quad (i = 1, 2).$$

Based on the Theorem 2.6, we consider the following two-parameter $(\lambda_1 > 0 < \lambda_2)$ problem

$$\begin{cases} [(-u'(r))^N]' = \lambda_1 N r^{N-1} f_1(r, u, v), & r \in (0, R), \\ [(-v'(r))^N]' = \lambda_2 N r^{N-1} f_2(r, u, v), & r \in (0, R), \\ u(r) > 0, \quad v(r) > 0, & r \in (0, R), \\ u'(0) = u(R) = v(R) = v'(0) = 0. \end{cases} \tag{2.8}$$

Theorem 2.9. *Assume that (H_f) holds. Then there exists $\lambda_1^* > 0 < \lambda_2^*$ such that for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem (2.8) has a nontrivial solution.*

In addition, if either (H_f^1) or (H_f^2) is satisfied, then problem (2.8) has at least one positive solution.

Proof In Theorem 2.6, we replace $\lambda_i f_i$ instead of f_i $(i = 1, 2)$, then it is worth noting that for any

$$\lambda_i > \frac{\big(\frac{\alpha}{R-b}\big)^N}{\int_0^b N\tau^{N-1} f_i(\tau, \alpha, \alpha)d\tau} := \lambda_i^* \quad (i = 1, 2),$$

(2.6) holds. Moreover, from Lemmas 2.7 and 2.8, we know that the nontrivial solution is positive. □

Corollary 2.10. *Assume that (H_f) holds. Then there exist constants $\lambda_1^* > 0 < \lambda_2^*$ such that for all $\lambda_1 > \lambda_1^*$ and $\lambda_2 > \lambda_2^*$, problem*

$$\begin{cases} \det D^2 u = \lambda_1 f_1(|x|, -u, -v) & \text{in } \mathcal{B}(R), \\ \det D^2 v = \lambda_2 f_2(|x|, -u, -v) & \text{in } \mathcal{B}(R), \\ u = v = 0 & \text{on } \partial\mathcal{B}(R) \end{cases} \tag{2.9}$$

has a nontrivial radial solution.

In addition, if either (H_f^1) or (H_f^2) is satisfied, then problem (2.9) has at least one radial solution (u, v) with both u and v are strictly increasing.

3. Sub- or superlinear nonlinearities near origin

In this section, we focus on the existence of positive solutions to problem (2.2) when f_1 (resp. f_2) with sub- or superlinear growth near origin with respect u (resp. v).

Theorem 3.1. *Let $f_i : [0, R] \times [0, \infty)^2 \to [0, \infty)$ $(i = 1, 2)$ be continuous and satisfy (H_f^1) (i). If $f_1(r, s, t)$ (resp. $f_2(r, s, t)$) is quai-monotone nondecreasing with respect to t (reap. s) and*

$$\lim_{s \to 0^+} \frac{f_1(r, s, 0)}{s^N} = +\infty \quad \textit{uniformly with} \quad r \in [0, R], \tag{3.1}$$

$$\lim_{t \to 0^+} \frac{f_2(r, 0, t)}{t^N} = +\infty \quad \textit{uniformly with} \quad r \in [0, R], \tag{3.2}$$

then problem (2.2) admits a solution (u,v) with $u \geq 0 \leq v$ and either u or v is positive and strictly decreasing.

In addition, if (H_f^1) (ii) holds, then problem (2.2) admits a positive solution (u,v) with both u and v are strictly decreasing.

Proof We first verify that there exists a constant $d_1 \in (0, MR(R+1))$ such that problem

$$\begin{cases} [(-u'(r))^N]' = Nr^{N-1}[f_1(r,u,v)+\mu], & r \in (0,R), \\ [(-v'(r))^N]' = Nr^{N-1}[f_2(r,u,v)+\mu], & r \in (0,R), \\ u(r) > 0, \quad v(r) > 0, & r \in (0,R), \\ u'(0) = u(R) = v(R) = v'(0) = 0 \end{cases} \tag{3.3}$$

has at most the trivial solution in $\overline{B}(d_1)$ for all $\mu \in [0,1]$.

Suppose on the contrary that there exist sequences $\{\mu_k\} \subset [0,1]$, $\{(u_k,v_k)\} \subset P \setminus \{(0,0)\}$, $\|(u_k,v_k)\| \to 0$, such that (u_k,v_k) is a nontrivial solution of (3.3) with $\mu = \mu_k$, for all $k \in \mathbb{N}$. From Lemma 2.7, we know that either u_k or v_k is positive and strictly decreasing. Without loss of generality, we may assume that u_k is positive for all $k \in \mathbb{N}$ (when v_k is positive, the conclusion also holds).

Choose a constant $m > 0$ such that

$$m > 9R^{-2}. \tag{3.4}$$

Then, it follows from (3.1) that we can seek a $k_0 \in \mathbb{N}$ such that

$$f_1(r,u_k(r),0) \geq (mu_k(r))^N \quad \text{for all} \quad r \in [0,R] \quad \text{and} \quad k \geq k_0. \tag{3.5}$$

Further, integrating the first equation in (3.3) over $[0,r]$ with $u = u_k$, $v = v_k$, $\mu = \mu_k$, using (3.5) and the fact that $f_1(r,s,t)$ is quai-monotone nondecreasing with respect to t, we have

$$\begin{aligned} \Big(-u_k'(r)\Big)^N &= \int_0^r N\tau^{N-1}[f_1(\tau,u_k(\tau),v_k(\tau))+\mu_k]d\tau \\ &\geq \int_0^r N\tau^{N-1} f_1(\tau,u_k(\tau),0)d\tau \\ &\geq \int_0^r N\tau^{N-1}(mu_k(\tau))^N d\tau \\ &\geq m^N \int_0^r N\tau^{N-1}u_k^N(\tau)d\tau, \end{aligned}$$

that is

$$-u_k'(r) \geq m\Big(\int_0^r N\tau^{N-1}u_k^N(\tau)d\tau\Big)^{\frac{1}{N}}.$$

Integrating the above inequality on $[\frac{R}{3}, \frac{2R}{3}]$ one obtains

$$u_k(\tfrac{R}{3}) - u_k(\tfrac{2R}{3}) \geq \int_{\frac{R}{3}}^{\frac{2R}{3}} m\Big(\int_0^r N\tau^{N-1}u_k^N(\tau)d\tau\Big)^{\frac{1}{N}} dr.$$

Then, combining the facts that u_k is strictly decreasing on $[0,R]$ and $u_k > 0$ on $[0,R)$, we obatin

$$\begin{aligned} u_k(\tfrac{R}{3}) &\geq \int_{\frac{R}{3}}^{\frac{2R}{3}} m\Big(\int_0^r N\tau^{N-1}u_k^N(\tau)d\tau\Big)^{\frac{1}{N}} dr \\ &\geq \int_{\frac{R}{3}}^{\frac{2R}{3}} \frac{1}{3}mRu_k(\tfrac{R}{3})dr \\ &\geq \frac{1}{9}mR^2u_k(\tfrac{R}{3}), \end{aligned}$$

it derives that

$$m \leq 9R^{-2},$$

contradicts with (3.4). Note that (3.3) has no solution in $\overline{B}(d_1)$ for any $\mu \in [0,1]$.

Let $\mathcal{N}_{f+\mu}$ be the fixed point operator associated to (3.3). We consider the compact homotopy $\mathcal{H} : [0,1] \times \overline{B}(d_1) \to P$ given as follows

$$\mathcal{H}(\mu,(u,v)) = \mathcal{N}_{f+\mu}(u,v).$$

Further, for all $(\mu,(u,v)) \in [0,1] \times \partial B(d_1)$, we have $(u,v) \neq \mathcal{H}(\mu,(u,v))$ holds. It follows from the invariance under homotopy of the Leray-Schauder degree that

$$d_{LS}(I - \mathcal{H}(0,\cdot), B(d_1), 0) = d_{LS}(I - \mathcal{H}(1,\cdot), B(d_1), 0).$$

If $d_{LS}(I - \mathcal{H}(1,\cdot), B(d_1), 0) \neq 0$, then there exists $(u,v) \in B(d_1)$ with $\mathcal{H}(1,(u,v)) = (u,v)$, a contradiction.

Consequently,

$$d_{LS}(I - \mathcal{H}(1,\cdot), B(d_1), 0) = 0.$$

Combining this with Lemma 2.1, we have

$$d_{LS}(I - \mathcal{N}_f, B_{MR(R+1)} \setminus \overline{B}(d_1), 0) = 1.$$

Therefore, (2.2) admits a positive solution $(u,v) \in B_{MR(R+1)} \setminus \overline{B}(d_1)$. □

Remark 3.2. *It is known that if $f : [0,R] \times [0,\infty) \to [0,\infty)$ is continuous, $f(r,s) > 0$ for all $(r,s) \in (0,R] \times (0,\infty)$ and*

$$\lim_{s\to 0^+} \frac{f(r,s)}{s^N} = +\infty \quad \textit{uniformly with} \quad r \in [0,R],$$

then the following boundary value problem

$$\begin{cases} [(-u'(r))^N]' = Nr^{N-1} f(r,u), \\ u'(0) = u(R) = 0 \end{cases} \tag{3.6}$$

has a positive solution.

Corollary 3.3. *Assume that $f_i : [0,R] \times [0,\infty)^2 \to [0,\infty) (i = 1,2)$ are continuous and with (H_f^1) (i). If $f_1(r,s,t)$ (resp. $f_2(r,s,t)$) is quai-monotone nondecreasing with respect to t (resp. s) and (3.1), (3.2) hold, then the system (1.1) admits a radial solution (u,v) with either u or v is strictly increasing. In addition, if (H_f^1) (ii) holds, then problem (1.1) admits a radial solution (u,v) with both u and v are strictly increasing.*

Theorem 3.4. *Assume that $f_i : [0,R] \times [0,\infty)^2 \to [0,\infty)$ $(i = 1,2)$ are continuous and with (H_f^2). If there exists some $l > 0$ such that either*

$$\lim_{s\to 0^+} \frac{f_1(r,s,t)}{s^N} = 0 \quad \textit{uniformly with} \quad r \in [0,R], \quad t \in [0,l] \tag{3.7}$$

or

$$\lim_{t\to 0^+} \frac{f_2(r,s,t)}{t^N} = 0 \quad \textit{uniformly with} \quad r \in [0,R], \quad s \in [0,l], \tag{3.8}$$

then there exists $\rho_0 > 0$ such that

$$d_{LS}(I - \mathcal{N}_f, B(\rho), 0) = 1, \quad 0 < \rho \leq \rho_0.$$

Proof Let $0 < \varepsilon < \frac{2}{R^2}$. Assume that (3.7) holds (similar reasoning when (3.8) holds), then there exists $s_\varepsilon > 0$ such that for all $s \in (0, s_\varepsilon)$,

$$f_1(r,s,t) \leq (\varepsilon s)^N, \quad r \in [0,R], \quad t \in [0,l]. \tag{3.9}$$

Consider the compact homotopy

$$\mathcal{H} : [0,1] \times P \to P, \quad \mathcal{H}(t,u,v) = t\mathcal{N}_f(u,v).$$

We will show that there exists $\rho_0 > 0$ such that

$$(u,v) \neq \mathcal{H}(t,u,v), \quad (t,u,v) \in [0,1] \times (\overline{B}_{\rho_0} \setminus \{(0,0)\}).$$

By contradiction, we assume that

$$(u_k, v_k) = t_k \mathcal{N}_f(u_k, v_k)$$

with $t_k \in [0,1]$, $(u_k, v_k) \in P \setminus \{(0,0)\}$ and $\|(u_k, v_k)\| \to 0$ for all $k \in \mathbb{N}$. From Lemma 2.8, we know that both u_k and v_k are strictly positive on $[0, R)$. We assume that $\|u_k\|_\infty \le s_\varepsilon$ and $\|v_k\|_\infty \le l$ for all $k \in \mathbb{N}$. It follows from (3.9) that

$$f_1(r, u_k(r), v_k(r)) \le (\varepsilon \|u_k\|_\infty)^N, \quad r \in [0, R], \quad k \in \mathbb{N}.$$

For any $k \in \mathbb{N}$, we obtain

$$\begin{aligned}\|u_k\|_\infty &= \int_0^R \Big(\int_0^t N\tau^{N-1} f_1(\tau, u_k(\tau), v_k(\tau)) d\tau \Big)^{\frac{1}{N}} dt \\ &\le \int_0^R \Big(\int_0^t N\tau^{N-1} (\varepsilon \|u_k\|_\infty)^N d\tau \Big)^{\frac{1}{N}} dt \\ &= \frac{1}{2} \varepsilon R^2 \|u_k\|_\infty,\end{aligned}$$

by passing with $k \to \infty$, we obtain $\varepsilon \ge 2R^{-2}$, which is a contradiction. Then it follows from the invariance under homotopy of the fixed point index that

$$d_{LS}(I - \mathcal{N}_f, B(\rho), 0) = d_{LS}(I, B(\rho), 0) = 1, \quad 0 < \rho \le \rho_0.$$

□

Corollary 3.4. *Assume that $f_i : [0, R] \times [0, \infty)^2 \to [0, \infty)$ $(i = 1, 2)$ are continuous and with (H_f^2). If (3.7) and (3.8) hold, (2.2) admits a positive solution. Further, the system (1.1) admits a radial solution (u, v).*

4. Lower and upper solutions; degree estimations

In this section, we get some degree estimates by using the lower and upper solutions method, and further obtain the existence of solutions for problem (2.2). First, we give the definitions of upper and lower solutions, respectively.

A lower solution of (2.2) is a couple of nonnegative functions $(\alpha_u, \alpha_v) \in C^1 \times C^1$ with $\|\alpha_u'\|_\infty \le MR$, $\|\alpha_v'\|_\infty \le MR$, and $r \mapsto (u'(r))^N$, $r \mapsto (v'(r))^N$ are of class C^1 on $[0, R]$ and satisfies

$$\begin{cases} [(-\alpha_u'(r))^N]' \ge N r^{N-1} f_1(r, \alpha_u, \alpha_v), \\ [(-\alpha_v'(r))^N]' \ge N r^{N-1} f_2(r, \alpha_u, \alpha_v), \\ \alpha_u'(0) = \alpha_u(R) = \alpha_v(R) = \alpha_v'(0) = 0. \end{cases} \tag{4.1}$$

An upper solution of (2.2) is a couple of nonnegative functions $(\beta_u, \beta_v) \in C^1 \times C^1$ with $\|\beta_u'\|_\infty \le MR$, $\|\beta_v'\|_\infty \le MR$, and the mappings $r \mapsto (u'(r))^N$, $r \mapsto (v'(r))^N$ are of class C^1 on $[0, R]$ and satisfies

$$\begin{cases} [(-\beta_u'(r))^N]' \le N r^{N-1} f_1(r, \beta_u, \beta_v), \\ [(-\beta_v'(r))^N]' \le N r^{N-1} f_2(r, \beta_u, \beta_v), \\ \beta_u'(0) = \beta_u(R) = \beta_v(R) = \beta_v'(0) = 0. \end{cases} \tag{4.2}$$

Let

$$\mathcal{D}_{(\alpha,\beta)} := \{(u, v) \in P : \alpha_u \le u \le \beta_u, \alpha_v \le v \le \beta_v\}.$$

Theorem 4.1. *Assume that (2.2) has a lower solution (α_u, α_v) and an upper solution (β_u, β_v) such that $\alpha_u(r) \le \beta_u(r)$, $\alpha_v(r) \le \beta_v(r)$ for all $r \in [0, R]$ and $f_1(r, s, t)$ (resp. $f_2(r, s, t)$) is quasi-monotone nondecreasing with respect to t (resp. s). Then,*

(i) problem (2.2) has always a solution $(u, v) \in \mathcal{D}_{(\alpha,\beta)}$;

(ii) if (2.2) has a unique solution $(u_0, v_0) \in \mathcal{D}_{(\alpha,\beta)}$ and there exists $\rho_0 > 0$ such that $\overline{B}((u_0, v_0), \rho_0) \subset \mathcal{D}_{\alpha,\beta}$, then

$$d_{LS}(I - \mathcal{N}_f, B((u_0, v_0), \rho), 0) = 1 \quad \textit{for all} \quad 0 < \rho \le \rho_0.$$

Proof (i) Define two new continuous functions $\Gamma_i : [0,R]\times[0,\infty)^2 \to [0,\infty)$ $(i=1,2)$ as follows

$$\Gamma_1(r,s,t) = f_1(r,\gamma_1(r,s),\gamma_2(r,t)) - s + \gamma_1(r,s)$$

and

$$\Gamma_2(r,s,t) = f_2(r,\gamma_1(r,s),\gamma_2(r,t)) - t + \gamma_2(r,t),$$

where γ_i given as

$$\gamma_1(r,s) = \max\{\alpha_u(r), \min\{s,\beta_u(r)\}\}, \quad \gamma_2(r,t) = \max\{\alpha_v(r), \min\{t,\beta_u(r)\}\}.$$

Now, we consider the following new problem

$$\begin{cases} [(-u'(r))^N]' = Nr^{N-1}\Gamma_1(r,u,v), & r\in(0,R), \\ [(-v'(r))^N]' = Nr^{N-1}\Gamma_2(r,u,v), & r\in(0,R), \\ u(r)>0, \quad v(r)>0, & r\in(0,R), \\ u'(0)=u(R)=v(R)=v'(0)=0. \end{cases} \tag{4.3}$$

It follows from Lemma 2.2 that problem (4.3) has at least one solution. Now, we show that if (u,v) is a solution of (4.3), then for all $r\in[0,R]$, $(u(r),v(r))\subset\mathcal{D}_{\alpha,\beta}$. We only prove that $\alpha_u\le u$ on $[0,R]$, the remainder can be obtained analogously.

By contradiction, we suppose that there exists $r_0\in[0,R]$ such that

$$\max_{[0,R]}(\alpha_u - u) = \alpha_u(r_0) - u(r_0) > 0. \tag{4.4}$$

If $r_0\in(0,R)$, then $\alpha_u'(r_0)=u'(r_0)$ and there exists a sequence $\{r_k\}\subset(0,r_0)$ converging to r_0 such that $\alpha_u'(r_k)-u'(r_k)\ge 0$. Therefore, $\alpha_u'<0$ and $u'<0$ imply the following inequality

$$\Big(-\alpha_u'(r_k)\Big)^N - \Big(-\alpha_u'(r_0)\Big)^N \le \Big(-u'(r_k)\Big)^N - \Big(-u'(r_0)\Big)^N$$

holds. Further, it derives that

$$\Big[(-\alpha_u'(r))^N\Big]'_{r=r_0} \ge \Big[(-u'(r))^N\Big]'_{r=r_0}.$$

Hence, from the facts (α_u,α_v) is a lower solution of (2.2) and f_1 is quasi-monotone nondecreasing with respect to t, we derive that

$$\begin{aligned} \Big[(-\alpha_u'(r))^N\Big]'_{r=r_0} &\ge \Big[(-u'(r))^N\Big]'_{r=r_0} = Nr_0^{N-1}\Gamma_1(r_0,u(r_0),v(r_0)) \\ &= Nr_0^{N-1}[f_1(r_0,\gamma_1(r_0,u(r_0)),\gamma_2(r_0,v(r_0))) - u(r_0) + \gamma_1(r_0,u(r_0))] \\ &= Nr_0^{N-1}[f_1(r_0,\alpha_u(r_0),\gamma_2(r_0,v(r_0))) - u(r_0) + \alpha_u(r_0)] \\ &> Nr_0^{N-1}f_1(r_0,\alpha_u(r_0),\gamma_2(r_0,v(r_0))) \\ &\ge Nr_0^{N-1}f_1(r_0,\alpha_u(r_0),\alpha_v(r_0)) \\ &\ge \Big[(-\alpha_u'(r))^N\Big]'_{r=r_0}, \end{aligned}$$

which is a contradiction.

If $r_0=R$, then $\alpha_u(R)-u(R)>0$ can be obtained from (4.4), which is inconsistent with $\alpha_u(R)=u(R)=0$.

Finally, if $r_0=0$, then there exists a sufficiently small $\varepsilon_0>0$ such that for all $r_1\in(0,\varepsilon_0]$, we have

$$\alpha_u(r)-u(r)>0, \quad \text{and} \quad \alpha_u'(r_1)-u'(r_1)\le 0, \quad r\in[0,r_1].$$

It derives that

$$\Big(-\alpha_u'(r_1)\Big)^N \ge \Big(-u'(r_1)\Big)^N.$$

Based on the facts that (α_u, α_v) is a lower solution of (2.2) and f_1 is quasi-monotone nondecreasing with respect to t, we integrate the first equation of problem (4.3) from 0 to r_1 and obtain

$$\begin{aligned}\Big(-u'(r_1)\Big)^N &= \int_0^{r_1} Nr^{N-1}\Big[f_1(r,\gamma_1(r,u(r)),\gamma_2(r,v(r))) - u(r) + \gamma_1(r,u(r))\Big]dr\\ &> \int_0^{r_1} Nr^{N-1} f_1(r,\alpha_u(r),\gamma_2(r,v(r)))dr\\ &\geq \int_0^{r_1} Nr^{N-1} f_1(r,\alpha_u(r),\alpha_v(r))dr\\ &\geq \int_0^{r_1} \Big[\Big(-\alpha_u'(r)\Big)^N\Big]'dr\\ &= \Big(-\alpha_u'(r_1)\Big)^N,\end{aligned}$$

which is a contradiction. Consequently, for all $r \in [0,R]$, $\alpha_u(r) \leq u(r)$ holds.
(ii) Let $\mathcal{N}_\Gamma : P \to P$ be the fixed point operator associated with problem (4.3). From Lemma 2.2 and Lemma 4.1 (i), it follows that $\mathcal{N}_\Gamma = \mathcal{N}_f$ on $\mathcal{D}_{(\alpha,\beta)}$, and all fixed points (u,v) of $\mathcal{N}_\Gamma$ are contained in $\mathcal{D}_{(\alpha,\beta)}$, they are also fixed points of $\mathcal{N}_f$. Hence, (u_0,v_0) is the unique fixed point of $\mathcal{N}_\Gamma$.

Therefore, for sufficiently large d,

$$(I-\mathcal{N}_\Gamma)(\overline{B}(d)\setminus B((u_0,v_0),\rho_0)) \neq (0,0).$$

It follows from Lemma 2.2 and $\mathcal{N}_\Gamma = \mathcal{N}_f$ on $\mathcal{D}_{\alpha,\beta} \supset \overline{B}((u_0,v_0),\rho_0)$ that

$$d_{LS}(I-\mathcal{N}_\Gamma, B((u_0,v_0),\rho),0) = d_{LS}(I-\mathcal{N}_\Gamma, B(d),0) = 1, \quad 0<\rho\leq\rho_0.$$

□

5. Non-existence, existence and multiplicity for a Lane-Emden system

In this section, we consider the Lane-Emden system with Monge-Ampère operator

$$\begin{cases} \det D^2u = \lambda_1\nu_1(|x|)(-u)^{p_1}(-v)^{q_1} = 0 & \text{in } \mathcal{B}(R),\\ \det D^2v = \lambda_2\nu_2(|x|)(-u)^{p_2}(-v)^{q_2} = 0 & \text{in } \mathcal{B}(R),\\ u_{\partial\mathcal{B}(R)} = 0 = v_{\partial\mathcal{B}(R)}. \end{cases} \tag{5.1}$$

At first, we make the following hypothesis:

(H) The functions $\nu_1,\nu_2 : [0,R] \to [0,\infty)$ are continuous with $\nu_1(r) > 0 < \nu_2(r)$ for all $r \in (0,R]$, $0 < q_1, p_2 < \infty$ and $N < p_1, q_2 < \infty$.

We try to deal with the following system

$$\begin{cases} [(-u'(r))^N]' = Nr^{N-1}\lambda_1\nu_1(r)u^{p_1}v^{q_1}, & r\in(0,R),\\ [(-v'(r))^N]' = Nr^{N-1}\lambda_2\nu_2(r)u^{p_2}v^{q_2}, & r\in(0,R),\\ u(r)>0,\quad v(r)>0, & r\in(0,R),\\ u'(0)=u(R)=v(R)=v'(0)=0. \end{cases} \tag{5.2}$$

Let $\mathcal{B}(\rho)$ be the circle in $\mathbb{R}^2$ entered at the origin with radius ρ. Denote

$$\overline{\mathcal{B}}_0(\rho) = \overline{\mathcal{B}}(\rho)\cap([0,\infty)\times[0,\infty)) \tag{5.3}$$

and

$$\Theta := \{(\lambda_1,\lambda_2) : \lambda_1,\lambda_2 > 0 \text{ and (5.2) has at least one positive solution}\},$$

it is easy to get that Θ is nonempty and unbounded in both directions of axes $0\lambda_1$ and $0\lambda_2$.

Lemma 5.1. *Assume that (H) holds. Then, the following are true:*

(i) there exist constants $\Lambda_1,\Lambda_2 > 0$ such that $\Theta \subset [\Lambda_1,\infty)\times[\Lambda_2,\infty)$ and for all $(\lambda_1,\lambda_2) \in (0,\infty)^2 \setminus ([\Lambda_1,\infty)\times[\Lambda_2,\infty))$, problem (5.2) has only the trivial solution;

(ii) if $(\overline{\lambda}_1, \overline{\lambda}_2) \in \Theta$, then $[\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty) \subset \Theta$;

(iii) if $(\overline{\lambda}_1, \overline{\lambda}_2) \in \Theta$, then for all $(\lambda_1, \lambda_2) \in (\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty)$, problem (5.2) has at least two positive solutions.

Proof (i) Let $\lambda_1, \lambda_2 > 0$. From Lemma 2.2, there exists a sufficiently large constant ρ such that problem (5.2) has at least one positive solution in $\overline{\mathcal{B}}_0(\rho)$. Let $(u,v) \in \overline{\mathcal{B}}_0(\rho)$ be a positive solution of (5.2). It follows from Lemma 2.8 that u and v are both strictly decreasing.

We denote $M_i := \max_{[0,R]} \nu_i(r)$, $(i = 1, 2)$. Since u and v are strictly decreasing on $[0, R]$, we deduce that

$$\begin{aligned} -u'(r) &= \Big(\int_0^r N t^{N-1} \lambda_1 \nu_1(t) u^{p_1}(t) v^{q_1}(t) dt \Big)^{\frac{1}{N}} \\ &\le \Big(\int_0^r N t^{N-1} \lambda_1 M_1 u^{p_1}(0) v^{q_1}(0) dt \Big)^{\frac{1}{N}} \\ &\le r \Big(\lambda_1 M_1 u^{p_1}(0) v^{q_1}(0) \Big)^{\frac{1}{N}}. \end{aligned}$$

Moreover,

$$u(0) \le \frac{1}{2} R^2 \Big(\lambda_1 M_1 u^{p_1}(0) v^{q_1}(0) \Big)^{\frac{1}{N}} \tag{5.4}$$

and

$$v(0) \le \frac{1}{2} R^2 \Big(\lambda_2 M_2 u^{p_2}(0) v^{q_2}(0) \Big)^{\frac{1}{N}}. \tag{5.5}$$

From Lemma 2.1, we have for any $(u,v) \in \overline{\mathcal{B}}_0(\rho)$, $0 < u(0), v(0) < MR^2$. Combining this with $p_1, q_2 > N$, we know

$$\lambda_i \ge \frac{(2M^2)^N}{M_i (MR)^{2p_i + 2q_i}} > 0, \quad (i = 1, 2). \tag{5.6}$$

Now, we consider the two nonempty sets

$$\Theta_1 := \{\lambda_1 > 0 : \exists \lambda_2 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \Theta\},$$

$$\Theta_2 := \{\lambda_2 > 0 : \exists \lambda_1 > 0 \text{ such that } (\lambda_1, \lambda_2) \in \Theta\}$$

and define

$$(0 <) \Lambda_i := \inf \Theta_i (< +\infty), \quad i = 1, 2.$$

Therefore, it follows that $\Theta \subset [\Lambda_1, \infty) \times [\Lambda_2, \infty)$. In addition, according to Lemma 2.8, we know that for all $(\lambda_1, \lambda_2) \in (0, \infty)^2$, (5.2) has at least one nontrivial solution, then, we obtain that for all $(\lambda_1, \lambda_2) \in (0, \infty)^2 \setminus ([\Lambda_1, \infty) \times [\Lambda_2, \infty))$, problem (5.2) has only the trivial solution.

(ii) Let $(\lambda_1^0, \lambda_2^0) \in [\overline{\lambda}_1, +\infty) \times [\overline{\lambda}_2, +\infty)$ and $(\overline{u}, \overline{v})$ be a positive solution of problem (5.2) with $\lambda_1 = \overline{\lambda}_1$ and $\lambda_2 = \overline{\lambda}_2$. Then $(\overline{u}, \overline{v})$ is a lower solution of (5.2) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. Combine the facts that $\lambda_1^0, \lambda_2^0 > 0$ and $(\overline{u}, \overline{v})$ is positive, we obtain $(\lambda_1^0, \lambda_2^0) \in \Theta$.

(iii) From (ii), we have that $(\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty) \subset \Theta$. Set $(\lambda_1^0, \lambda_2^0) \in (\overline{\lambda}_1, +\infty) \times (\overline{\lambda}_2, +\infty)$. Next, we will show that when $\lambda_1 = \lambda_1^0$, $\lambda_2 = \lambda_2^0$, problem (5.2) has a second positive solution. For this, we remain let $(\overline{u}, \overline{v})$ be the lower solution as we constructed before.

At first, let (u_0, v_0) be a positive solution of (5.2) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$ such that $(u_0, v_0) \in \mathcal{D}_{(\overline{u},\overline{v})} := \{(u,v) \in K : \overline{u} \le u, \overline{v} \le v\}$. Now, we claim that there exists $\varepsilon > 0$ such that $\overline{B}((u_0, v_0), \varepsilon) \subset \mathcal{D}_{(\overline{u},\overline{v})}$. We only need to show that for any $(u,v) \in \overline{B}((u_0, v_0), \varepsilon)$, we have $(u,v) \in \mathcal{D}_{(\overline{u},\overline{v})}$.

In fact, for all $r \in [0, \frac{R}{2}]$, we can deduce that

$$\begin{aligned} \overline{u}(r) &= \int_r^R \Big(\int_0^t N s^{N-1} \Big[\overline{\lambda}_1 \nu_1(s) \overline{u}^{p_1}(s) \overline{v}^{q_1}(s) \Big] ds \Big)^{\frac{1}{N}} dt \\ &< \int_r^R \Big(\int_0^t N s^{N-1} \Big[\lambda_1^0 \nu_1(s) u_0^{p_1}(s) v_0^{q_1}(s) \Big] ds \Big)^{\frac{1}{N}} dt \\ &= u_0(r). \end{aligned}$$

Similarly, we obtain $\overline{v}(r) < v_0(r)$ on $[0, \frac{R}{2}]$. Therefore, we can find $\varepsilon_1 > 0$ such that if $(u, v) \in K$, then

$$\|u - u_0\|_\infty \leq \varepsilon_1 \Rightarrow \overline{u} \leq u \quad \text{and} \quad \|v - v_0\|_\infty \leq \varepsilon_1 \Rightarrow \overline{v} \leq v \quad \text{on} \quad [0, \tfrac{R}{2}]. \tag{5.7}$$

On the other hand, for any $r \in [\frac{R}{2}, R]$, one obtains $u_0'(r) < \overline{u}'(r)$ and $v_0'(r) < \overline{v}'(r)$. Hence, there exists some $\varepsilon_2 \in (0, \varepsilon_1)$ such that if $(u, v) \in K$, then

$$\|u' - u_0'\|_\infty \leq \varepsilon_2 \Rightarrow \overline{u}' > u' \quad \text{and} \quad \|v' - v_0'\|_\infty \leq \varepsilon_2 \Rightarrow \overline{v}' > v' \quad \text{on} \quad [\tfrac{R}{2}, R].$$

Since

$$u(r) = -\int_r^R u'(s)ds > -\int_r^R \overline{u}'(s)ds = \overline{u}(r),$$

we have $u > \overline{u}$ on $[\frac{R}{2}, R)$. Analogously, $v > \overline{v}$ on $[\frac{R}{2}, R)$. In addition, we also have $u(R) = \overline{u}(R)$, which means that

$$\|u' - u_0'\|_\infty \leq \varepsilon_2 \Rightarrow \overline{u} \leq u \quad \text{and} \quad \|v' - v_0'\|_\infty \leq \varepsilon_1 \Rightarrow \overline{v} \leq v \quad \text{on} \quad [\tfrac{R}{2}, R]. \tag{5.8}$$

Consequently, $\overline{u} \leq u$ and $\overline{v} \leq v$ on $[0, R]$. Choose $\varepsilon \in (0, \varepsilon_2)$, then the claim holds.

Next, if problem (5.2) has a second solution involving in $\mathcal{D}_{(\overline{u}, \overline{v})}$, then the solution is nontrivial and the conclusion is obvious. Otherwise, by Lemma 4.1, we can deduce that

$$d_{LS}(I - \mathcal{N}_{(\lambda_1^0, \lambda_2^0)}, B((u_0, v_0), \rho_1), 0) = 1, \quad 0 < \rho_1 \leq \varepsilon,$$

where $\mathcal{N}_{(\lambda_1^0, \lambda_2^0)}$ stands for the fixed point operator associated to problem (5.2) with $\lambda_1 = \lambda_1^0$ and $\lambda_2 = \lambda_2^0$. Furthermore, from Lemma 2.1, we get

$$d_{LS}(I - \mathcal{N}_{(\lambda_1^0, \lambda_2^0)}, B(\rho_3), 0) = 1, \quad \rho_3 \geq MR(R+1),$$

and from Theorem 3.4, for sufficiently small $\rho_2 > 0$, we have

$$d_{LS}(I - \mathcal{N}_{(\lambda_1^0, \lambda_2^0)}, B(\rho_2), 0) = 1.$$

Let $\rho_1, \rho_2 > 0$ be sufficiently small be such that $\overline{B}((u_0, v_0), \rho_1) \cap \overline{B}(\rho_2) = \emptyset$ and $(\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_2)) \subset B(\rho_3)$. By the additivity and excision of the fixed point index, it can be obtained

$$d_{LS}\Big(I - \mathcal{N}_{(\lambda_1^0, \lambda_2^0)}, B(\rho_3) \setminus \big[\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_2)\big], 0\Big) = -1.$$

So, $\mathcal{N}_{(\lambda_1^0, \lambda_2^0)}$ has a fixed point $(u, v) \in B(\rho_3) \setminus \big[\overline{B}((u_0, v_0), \rho_1) \cup \overline{B}(\rho_2)\big]$. Combing this with the fact that $(\lambda_1^0, \lambda_2^0) \in \Theta$, we know that problem (5.2) has at least two positive solutions.

□

Now, for $\theta \in (0, \frac{\pi}{2})$, we set

$$\mathcal{L}(\theta) := \{\lambda > 0 : (\lambda \cos\theta, \lambda \sin\theta) \in \Theta\},$$

it is a nonempty set. We reconsider problem (5.2) in the following form

$$\begin{cases} [(-u'(r))^N]' = Nr^{N-1}\lambda\cos\theta\nu_1(r)u^{p_1}v^{q_1}, & r \in (0, R), \\ [(-v'(r))^N]' = Nr^{N-1}\lambda\sin\theta\nu_2(r)u^{p_2}v^{q_2}, & r \in (0, R), \\ u(r) > 0, \quad v(r) > 0, & r \in (0, R), \\ u'(0) = u(R) = v(R) = v'(0) = 0, \end{cases} \tag{5.9}$$

where $\lambda > 0$ is a real parameter.

Lemma 5.2. *There exists a continuous function $\Lambda : (0, \frac{\pi}{2}) \to (0, \infty)$ such that*

$$\lim_{\theta \to 0} \Lambda(\theta) \sin\theta - \Lambda_2 = 0 = \lim_{\theta \to \frac{\pi}{2}} \Lambda(\theta) \cos\theta - \Lambda_1, \tag{5.10}$$

and has the following results hold:

(i) $\Lambda(\theta) \in \mathcal{L}(\theta)$, $\theta \in (0, \frac{\pi}{2})$;

(ii) there exists a sufficiently large ρ such that for all $(\lambda_1, \lambda_2) \in (\Lambda(\theta)\cos\theta, \infty) \times (\Lambda(\theta)\sin\theta, \infty)$, $\theta \in (0, \frac{\pi}{2})$, system (5.2) has at least two positive solutions.

Proof Set

$$\Lambda(\theta) := \inf \mathcal{L}(\theta), \qquad \theta \in (0, \tfrac{\pi}{2}). \tag{5.11}$$

It follows from $\mathcal{L}(\theta) \neq \emptyset$, $\mathcal{L}(\theta) > 0$ and Lemma 5.1 (i) that $\Lambda(\theta) < \infty$.

(i) Let $\{\lambda^k\} \subset \mathcal{L}(\theta)$ be a decreasing sequence converging to $\Lambda(\theta)$ and $(u_k, v_k) \in \overline{\mathcal{B}}_0(\rho)$ ($\overline{\mathcal{B}}_0(\rho)$ is given in (5.3)) with $u_k > 0 < v_k$ on $[0, R)$ be such that

$$u_k = P \circ S \circ \left[\lambda^k \cos\theta \nu_1 u_k^{p_1} v_k^{q_1}\right],$$
$$v_k = P \circ S \circ \left[\lambda^k \sin\theta \nu_2 u_k^{p_2} v_k^{q_2}\right].$$

From Lemma 2.1 and Arzela-Ascoli Theorem, we know that there exists $(u, v) \in \overline{\mathcal{B}}_0(\rho)$ such that a sequence $\{(u_k, v_k)\}$ converges to (u, v) in $C \times C$ by the usual product topology. Hence, $u \geq 0 \leq v$ and

$$u = P \circ S \circ \left[\Lambda(\theta) \cos\theta \nu_1 u^{p_1} v^{q_1}\right],$$
$$v = P \circ S \circ \left[\Lambda(\theta) \sin\theta \nu_2 u^{p_2} v^{q_2}\right].$$

From (5.4) and (5.5), we obtain

$$u_k(0) \leq \tfrac{1}{2} R^2 \Big(\lambda^k \cos\theta M_1 u_k^{p_1}(0) v_k^{q_1}(0)\Big)^{\frac{1}{N}}$$

and

$$v_k(0) \leq \tfrac{1}{2} R^2 \Big(\lambda^k \sin\theta M_2 u_k^{p_2}(0) v_k^{q_2}(0)\Big)^{\frac{1}{N}}.$$

Furthermore, since $0 < u_k(0), v_k(0) < MR^2$, it imply that

$$u_k^{\frac{p_1 - N}{N}}(0) > \frac{2}{\left(\lambda^k \cos\theta M_1 M^{q_1}\right)^{\frac{1}{N}} R^{\frac{2q_1 + 2N}{N}}}$$

and

$$v_k^{\frac{q_2 - N}{N}}(0) > \frac{2}{\left(\lambda^k \sin\theta M_2 M^{p_2}\right)^{\frac{1}{N}} R^{\frac{2p_2 + 2N}{N}}}.$$

The fact $N < p_1, q_2 < \infty$ guarantees that there is a constant $c > 0$ such that for all k, $u_k(0), v_k(0) > c$ hold true. This leads to $u(0), v(0) \geq c$. Therefore, by Lemma 2.8, we have $u > 0 < v$ on $[0, R)$, which means that $\Lambda(\theta) \in \mathcal{L}(\theta)$.

(ii) This fact comes from Lemma 5.1 (iii). Firstly, we need to prove that Λ is continuous at each $\theta_0 \in (0, \frac{\pi}{2})$. Otherwise, we can find some $\varepsilon \in (0, \Lambda(\theta_0))$ such that for all sufficiently large $n \in \mathbb{N}$, there exists $\theta_n \in (\theta_0 - \frac{1}{n}, \theta_0 + \frac{1}{n}) \subset (0, \frac{\pi}{2})$ with

$$|\Lambda(\theta_n) - \Lambda(\theta_0)| \geq \varepsilon.$$

Assume that $\Lambda(\theta_n) - \Lambda(\theta_0) \geq \varepsilon$ for infinitely many $n \in \mathbb{N}$. Then for a subsequence of $\{\theta_n\}$, still denoted by $\{\theta_n\}$, we have

$$(\Lambda(\theta_n) - \tfrac{\varepsilon}{2}) \cos\theta_n \geq (\Lambda(\theta_0) + \tfrac{\varepsilon}{2}) \cos\theta_n$$

and

$$(\Lambda(\theta_n) - \tfrac{\varepsilon}{2}) \sin\theta_n \geq (\Lambda(\theta_0) + \tfrac{\varepsilon}{2}) \sin\theta_n.$$

In addition, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, we obtain

$$(\Lambda(\theta_0) + \tfrac{\varepsilon}{2}) \cos\theta_n > \Lambda(\theta_0) \cos\theta_0$$

and

$$(\Lambda(\theta_0) + \tfrac{\varepsilon}{2}) \sin\theta_n > \Lambda(\theta_0) \sin\theta_0.$$

Therefore, for any $n \geq n_0$, we get

$$(\Lambda(\theta_n) - \tfrac{\varepsilon}{2}) \cos\theta_n \geq (\Lambda(\theta_0) + \tfrac{\varepsilon}{2}) \cos\theta_n > \Lambda(\theta_0) \cos\theta_0$$

and

$$(\Lambda(\theta_n) - \tfrac{\varepsilon}{2}) \sin\theta_n \geq (\Lambda(\theta_0) + \tfrac{\varepsilon}{2}) \sin\theta_n > \Lambda(\theta_0) \sin\theta_0.$$

Further, by invoking these facts that $\Lambda(\theta_0) \in \mathcal{L}(\theta_0)$ and Lemma 5.1 (ii), we obtain

$$\big((\Lambda(\theta_n) - \tfrac{\varepsilon}{2})\cos\theta_n, (\Lambda(\theta_n) - \tfrac{\varepsilon}{2})\sin\theta_n\big) \in \Theta,$$

which means that $\Lambda(\theta_n) - \frac{\varepsilon}{2} \in \mathcal{L}(\theta_n)$, it contradicts the definition of $\Lambda(\theta_n)$. Analogously, since for infinitely many $n \in \mathbb{N}$, $\Lambda(\theta_n) - \Lambda(\theta_0) \leq -\varepsilon$, using the similarly method, a similar contradiction of the definition of $\Lambda(\theta_0)$ may arise. According to the Heine's theorem, Λ is continuous on $(0, \frac{\pi}{2})$.

Finally, for the fact stated by Lemma 5.1 (iii), we need to prove that (5.10) holds. Let $\{\theta_n\} \subset (0, \frac{\pi}{2})$ be a sequence with $\theta_n \to \frac{\pi}{2}(n \to \infty)$. Based on this, we have to prove that

$$\Lambda(\theta_n)\cos\theta_n \to \Lambda_1, \quad n \to \infty. \tag{5.12}$$

For this, we only need to show that any subsequence $\{\theta_{n_k}\} \subset \{\theta_n\}$ with

$$\Lambda(\theta_{n_k})\cos\theta_{n_k} \to \Lambda_1, \quad k \to \infty.$$

Since $\Lambda_1 = \inf \Sigma_1$, there exists a sequence $\{\lambda_1^k\} \subset \Theta_1$ with $\lambda_1^k \to \Lambda_1(k \to \infty)$. Since $\theta_n \to \frac{\pi}{2}(n \to \infty)$, using an inductive standard reasoning for Lemma 5.1 (ii), we can seek a sequence $\{r_k\} \subset (0, \infty)$ such that for any subsequence $\{\theta_{n_k}\} \subset \{\theta_n\}$ and for all $k \in \mathbb{N}$, we have

$$r_k \cos\theta_{n_k} = \lambda_1^k \tag{5.13}$$

and

$$(r_k \cos\theta_{n_k}, r_k \sin\theta_{n_k}) \in \Theta.$$

Recalling the definition of Λ, we know that $\Lambda(\theta_{n_k}) \leq r_k$, it follows that $\Lambda(\theta_{n_k})\cos\theta_{n_k} \leq r_k\cos\theta_{n_k}$. Furthermore, by the definition of Λ_1 and (5.13), we have

$$\Lambda_1 \leq \Lambda(\theta_{n_k})\cos\theta_{n_k} \leq r_k\cos\theta_{n_k} = \lambda_1^k \to \Lambda_1, \quad k \to \infty.$$

By the reduction principle, it means that $\Lambda(\theta_n)\cos\theta_n \to \Lambda_1$, $\theta_n \to \frac{\pi}{2}(n \to \infty)$. Meanwhile, it also can be proved that $\Lambda(\theta_n)\sin\theta_n \to \Lambda_2$ when $\theta_n \to 0(n \to \infty)$, that is (5.10) holds.

□

Theorem 5.3. *Assume that (H) holds. Then there exist* $\Lambda_1, \Lambda_2 > 0$ *and a continuous function* $\Lambda : (0, \frac{\pi}{2}) \to (0, \infty)$*, generating the curve*

$$(\Gamma) = \begin{cases} \lambda_1(\theta) = \Lambda(\theta)\cos\theta, \\ \lambda_2(\theta) = \Lambda(\theta)\sin\theta, \end{cases} \qquad \theta \in (0, \tfrac{\pi}{2})$$

such that

(i) $\Gamma \subset [\Lambda_1, +\infty) \times [\Lambda_2, +\infty)$*;*

(ii) the following asymptotic behaviors hold

$$\lim_{\theta \to \frac{\pi}{2}} \lambda_2(\theta) = +\infty = \lim_{\theta \to 0} \lambda_1(\theta), \tag{5.14}$$

$$\lim_{\theta \to 0} \lambda_2(\theta) - \Lambda_2 = 0 = \lim_{\theta \to \frac{\pi}{2}} \lambda_1(\theta) - \Lambda_1; \tag{5.15}$$

(iii) Γ *divides the first quadrant* $(0, +\infty) \times (0, +\infty)$ *into two disjoint sets* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *such that problem (5.1) has zero, at least one or at least two radial solutions when* $(\lambda_1, \lambda_2) \in \mathcal{O}_1$*,* $(\lambda_1, \lambda_2) \in \Gamma$ *or* $(\lambda_1, \lambda_2) \in \mathcal{O}_2$*, respectively.*

Proof Lemma 5.1 (i) guarantees the existence of Λ_1 and Λ_2, whereas Lemma 5.2 ensures the existence of the continuous function Λ.

(i) It can be proved by Lemma 5.1 (i) together with Lemma 5.2 (i);

(ii) Since $\Lambda(\theta) \geq \frac{\Lambda_1}{\cos\theta}$, which means that

$$\lim_{\theta \to \frac{\pi}{2}} \lambda_2(\theta) \geq \lim_{\theta \to \frac{\pi}{2}} \Lambda_1 \tan\theta = \infty.$$

Analogously, $\lim_{\theta \to 0} \lambda_1(\theta) = \infty$ follows from that $\Lambda(\theta) \geq \frac{\Lambda_2}{\sin\theta}$. And equalities (5.14) hold from Lemma 5.2;

(iii) From Lemmas 5.1 and 5.2, we know that the curve Γ divides the first quadrant into two disjoint unbounded open sets $\mathcal{O}_1$ and $\mathcal{O}_2$, and the set $\mathcal{O}_1$ is adjacent to the coordinate axes $0\lambda_1$ and

$0\lambda_2$, the curve Γ is asymptotically close to two straight lines parallel to the coordinate axes $0\lambda_1$, $0\lambda_2$. From this, we know that the system (5.1) has zero, at least one or two radial solutions according to $(\lambda_1, \lambda_2) \in \mathcal{O}_1$, $(\lambda_1, \lambda_2) \in \Gamma$ or $(\lambda_1, \lambda_2) \in \mathcal{O}_2$, respectively. $\square$

Based on the result of Theorem 5.3, we consider a more general system problem:

$$\begin{cases} \det D^2 u = \lambda_1 \nu_1(|x|) f_1(|x|, -u, -v) & \text{in } \mathcal{B}(R), \\ \det D^2 v = \lambda_2 \nu_2(|x|) f_2(|x|, -u, -v) & \text{in } \mathcal{B}(R), \\ u = v = 0 & \text{on } \partial\mathcal{B}(R), \end{cases} \tag{5.16}$$

where weight functions $\nu_1, \nu_2 \in C([0, R], [0, \infty))$ with $\nu_1(r) > 0 < \nu_2(r)$ for all $r \in (0, R]$, p_1, q_2 are nonnegative and q_1, p_2 are positive exponents, f_1 and $f_2 : [0, R] \times [0, \infty)^2 \to [0, \infty)$ are continuous.

Corollary 5.4. *Assume that $f_1(r, s, t)$, $f_2(r, s, t)$ are quasi-monotone nondecreasing with respect to both s and t and satisfy the additional condition:*

$(\widetilde{H}_f)$ *there exist constants $c > 0$, $p_1, q_2 > N$ and $q_1, p_2 > 0$ such that*

$$0 < f_1(r, s, t) \le c s^{p_1} t^{q_1},$$

$$0 < f_2(r, s, t) \le c s^{p_2} t^{q_2}$$

for all $s, t > 0$.

Then for problem (5.16), all results in Theorem 5.3 are still valid.

Acknowledgment

Supported by National Natural Science Foundation of China [Grant No. 11961060], The Key Project of Natural Sciences Foundation of Gansu Province [Grant No. 18JR3RA084] and Graduate Research Support Project of Northwest Normal University [Grant No. 2021KYZZ01032].

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