Stability Analysis of Interchange-Stable Plasma Sheet to $\vec{E} \times \vec{B}$ Shear Flow at Substorm Onset

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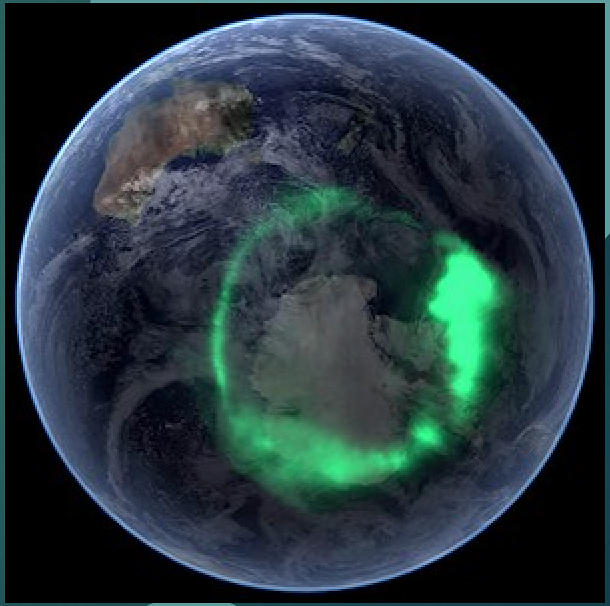
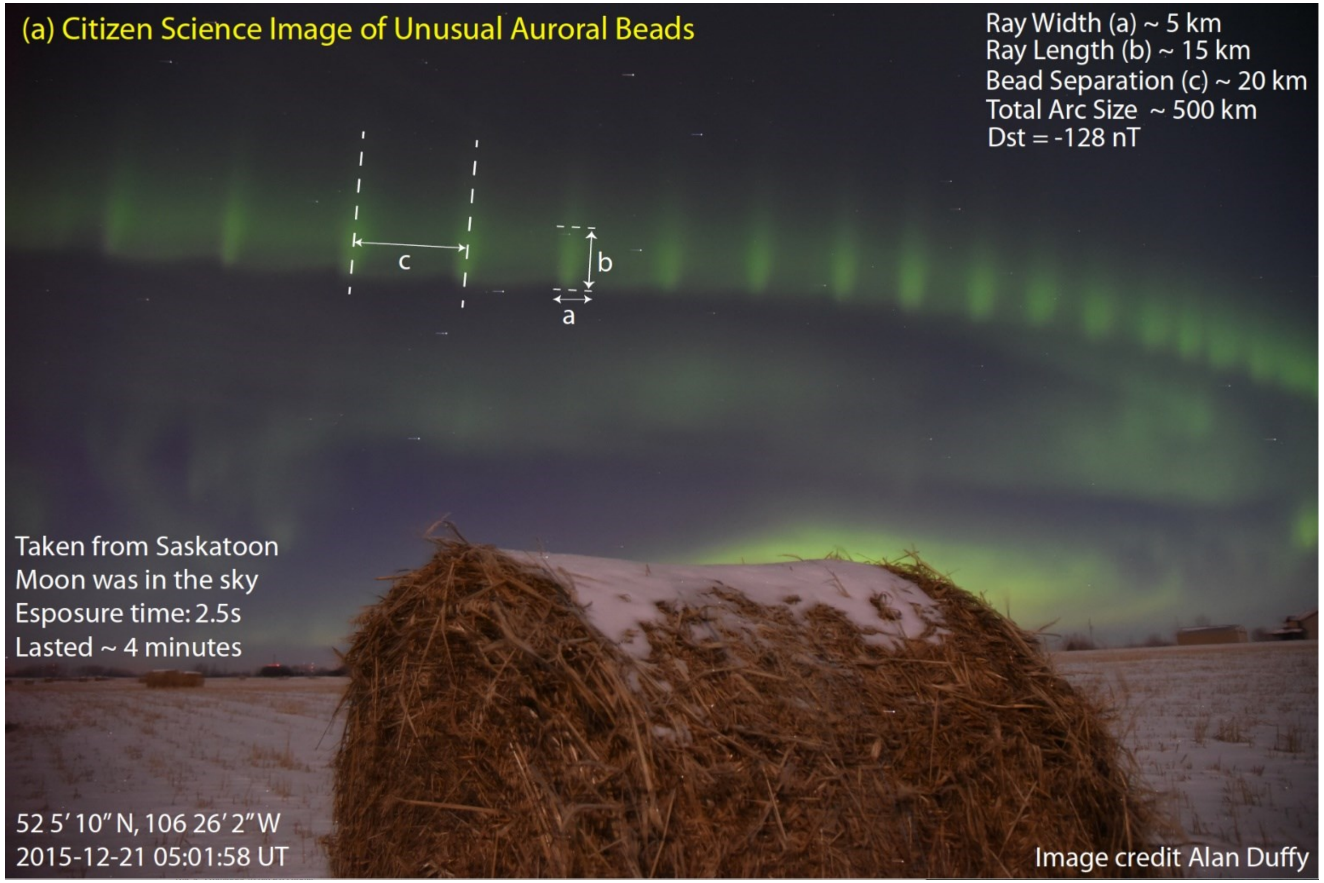


Auroral Beads

- Onset of magnetic substorms pervasively marked by auroral beads.
- Use bead structure to indirectly determine cause of substorm onset.

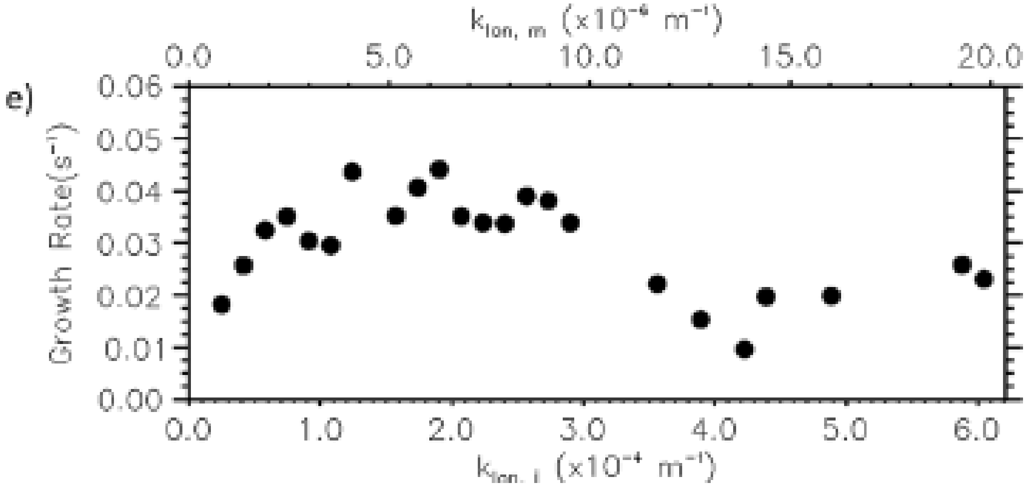


Kalmoni Strategy to Determine Cause of Substorm Onset

- Optical analysis of each substorm from ASI data
- Statistical analysis of many substorms
- Assume beads are the ionospheric projection of a magnetospheric instability
- Compare magnetospheric instability candidates with projected ionospheric signatures

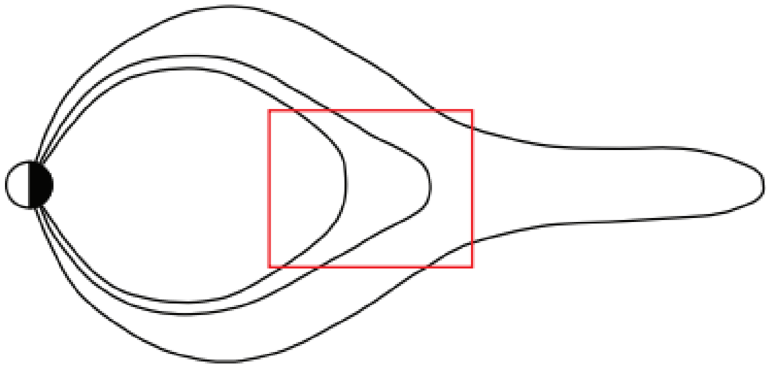
Optical Analysis

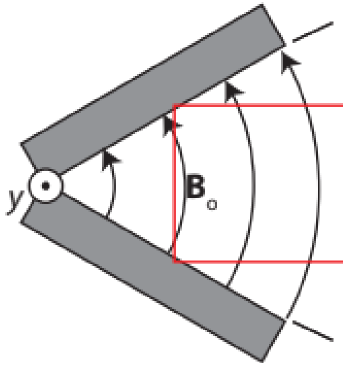
- Growth rates as function of wavenumber for determination of most unstable wavenumber (for comparison with theory)
  - Growth rates peak around $\gamma \sim 0.045\ s^{-1}$ at $k_{lon,i} = 2.0 * 10^{-4}\ m^{-1}$
  - Naively mapping to magnetosphere, we obtain $k_{lon,m} = 6.0 * 10^{-6}\ m^{-1}$



Equilibrium Wedge Model

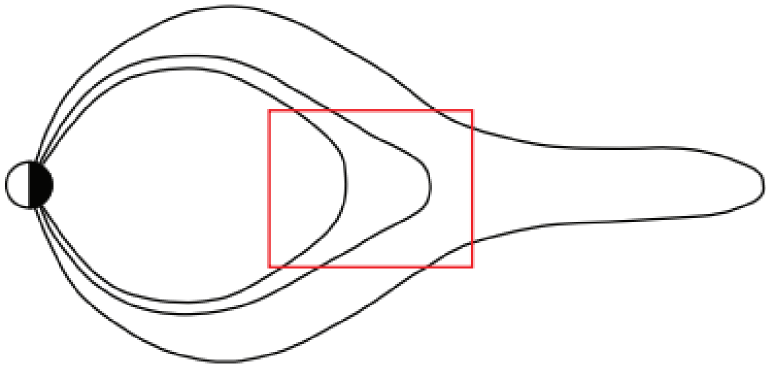
- Near-Earth nightside of magnetosphere, center of magnetosphere taken to be origin of cylindrical coordinate system (r, φ, y).
- Field lines are concentric circles, and pressure and field strength varies radially. $(P_0, B_0, v_0, K_0, \rho_0)$
- Adiabatic pressure dynamics $(K \coloneqq PV^{\Gamma})$ for flux tubes.
- Flux tube volume $V(r) = r\Delta\varphi / B(r)$.
- No field-aligned conductances
- Equilibrium pressure balance
- System in equilibrium, barring small perturbations which induce no angular motion.

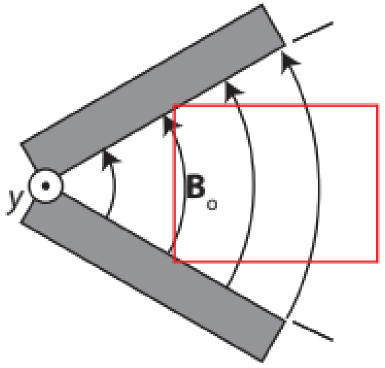




Derivation of Wedge Equation

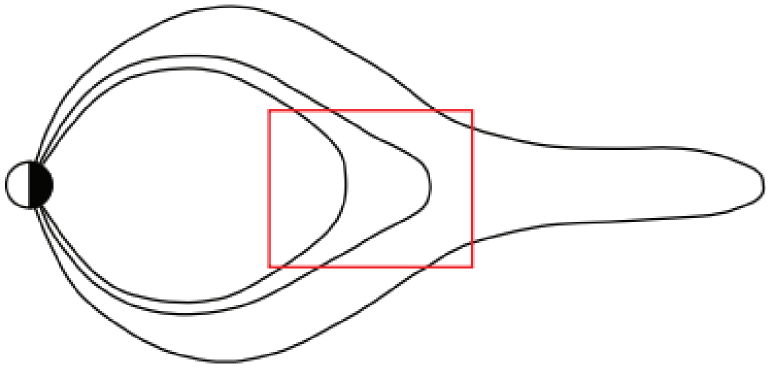
- Momentum equation and Faraday's (Ohm's) law:
  - $\frac{D}{Dt}(\rho\vec{v}) = -\nabla\left(P + \frac{B^2}{2\mu_0}\right) + \frac{\vec{B}\cdot\nabla\vec{B}}{\mu_0}$
  - $\frac{\partial\vec{B}}{\partial t} = \nabla\times(\vec{v}\times\vec{B})$
- Flux tube adiabaticity serves as closure:
  - $\frac{DK}{Dt} = \frac{D}{Dt}(PV^{\Gamma}) = 0 \rightarrow \frac{D}{Dt}\left(P\left(\frac{r}{B}\right)^{\Gamma}\right) = 0.$
- Definitions:
  - $\widetilde{\omega} := \omega - k_y v_0(r)$, $c_A^2 := B_0^2/\mu_0\rho_0$, $c_s := \Gamma P_0/\rho_0$, $c_f^2 = c_A^2 + c_s^2$.

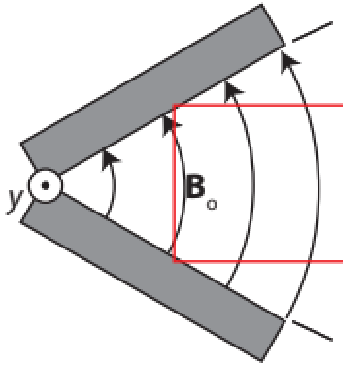




Derivation of Wedge Equation

- Use dawn-to-dusk waves as perturbative normal modes $\delta\blacksquare \sim e^{ik_y y - i\omega t}$.
- Combine to obtain second order decoupled radial velocity equation.
- Take low-frequency limit to obtain equation governing linear stage of shear flow-interchange instability.





Reduced Low-Frequency Wedge Equation

- Schrödinger Transformation:
  - $\delta u_r'' + V_{eff}(r, \widetilde{\omega}, k_y)\delta u_r = 0$
  - $\delta u_r \coloneqq Ce^{-\frac{1}{2}\int B_{lf}(r)dr}\delta v_r$

$$
\begin{aligned}
V_{eff}(r,\widetilde{\omega},k_y) \quad := \; & \boxed{\frac{k_y v_0''}{\widetilde{\omega}}} - 3\left(\frac{v_0'}{c_f}\right)^2 + \frac{k_y v_0'}{\widetilde{\omega}}\left(\frac{2\widetilde{\omega}^2 - k_y^2 c_s^2 + 3k_y^2 c_A^2}{k_y^2 c_f^2}\right)\frac{\rho_0'}{\rho_0} - \frac{k_y v_0'}{\widetilde{\omega}}\frac{1}{r} \\
& - \frac{3c_s^4 - 10c_s^2 c_A^2 + 3c_A^4}{c_f^4}\frac{1}{4r^2} + \frac{c_s^4 - 2c_s^2 c_A^2 - 3c_A^4}{c_f^4}\frac{1}{2r}\frac{\rho_0'}{\rho_0} - \frac{1}{2}\frac{\rho_0''}{\rho_0} \\
& + \frac{1}{4}\left(\frac{\rho_0'}{\rho_0}\right)^2 - \frac{k_y v_0'}{\widetilde{\omega}}\frac{\widetilde{\omega}^2}{k_y^2 c_f^2}\frac{c_s^2}{c_f^2}\frac{P_0'}{P_0} + \frac{2c_s^2 c_A^2}{c_f^4}\frac{1}{r}\frac{P_0'}{P_0} - \frac{k_y v_0'}{\widetilde{\omega}}\frac{\widetilde{\omega}^2}{k_y^2 c_f^2}\frac{2c_A^2}{c_f^2}\frac{B_0'}{B_0} \\
& - \frac{4c_s^2 c_A^2}{c_f^4}\frac{1}{r}\frac{B_0'}{B_0} + \boxed{\frac{k_y^2 c_f^2}{\widetilde{\omega}^2}\frac{2}{\Gamma r}\frac{c_s^2 c_A^2}{c_f^4}\frac{K_0'}{K_0}} - k_y^2
\end{aligned}
$$

Effective Potential and Fine Array

- First, potential is recast in Laurent series form (with $\omega \coloneqq \omega_r + i\gamma$):
  - $V(r,\omega,k_y) = \frac{1}{\left(v_0(r)-\frac{\omega}{k_y}\right)^2} V_{-2}(r) + \frac{1}{v_0(r)-\frac{\omega}{k_y}} V_{-1}(r) + V_0(r) + V_1(r)\frac{\omega}{k_y} - k_y^2$
- We need a finer array in order to appropriately resolve the resonances. The denominator is:
- $D(r) \coloneqq v_0(r) - \frac{\omega}{k_y} = v_0(r) - \frac{\omega_r + i\gamma}{k_y} \to v_0' \Delta r - i\frac{\gamma}{k_y}.$

, where the last step results from linearizing velocity near the point $\omega_r = k_y v_0$. So we obtain the following to resolve peaks of interest:

- $\Delta r = \frac{1}{N}\frac{\gamma}{k_y v_0'}, \quad N \gg 1$

Taylor-Goldstein Equation

- The main phenomenological part of our equation, and the part which dominates near the resonances, has the form of a Taylor-Goldstein equation. What I have been calling "pure Kelvin-Helmholtz" is the Rayleigh equation, which is a special case of the Taylor-Goldstein equation with no buoyancy. It can be written in the form:

$$\delta u_r''(r) + \left[W \frac{N^2(z)}{\left(v_0(r) - \frac{\omega}{k}\right)^2} - \frac{v_0''(r)}{v_0(r) - \frac{\omega}{k}} - k^2 \right] \delta u_r(r) = 0$$

where $N(z)$ is the buoyancy frequency.

Richardson's Instability Criterion

- We define the **Richardson's number**:

$$Ri(r) = W = \left(\frac{N(r)}{v_0'(r)}\right)^2$$

- A necessary condition for instability is that $W^* = Ri(r^*) = \min_r Ri(r) < \frac{1}{4}$ somewhere in the domain.

- Basically, in fluid mechanics, our resonances are called "critical layers" and singularities are called "critical points." In intuitive words, "sufficiently strong stratification always has a stabilizing effect."

- Proof of theorem can be found in Kundu's Fluid Mechanics Textbook.

Howards' Semicircle Theorem

- For any unstable eigenvalue ω, real and imaginary components must satisfy:

$$\left(\frac{\omega_r}{k} - \frac{v_{0,max} + v_{0,min}}{2}\right)^2 + \left(\frac{\gamma}{k}\right)^2 \leq \left(\frac{v_{0,max} - v_{0,min}}{2}\right)^2$$

Proof of main theorem can be found in Kundu's Fluid Mechanics Textbook.

- <u>Consequences:</u>

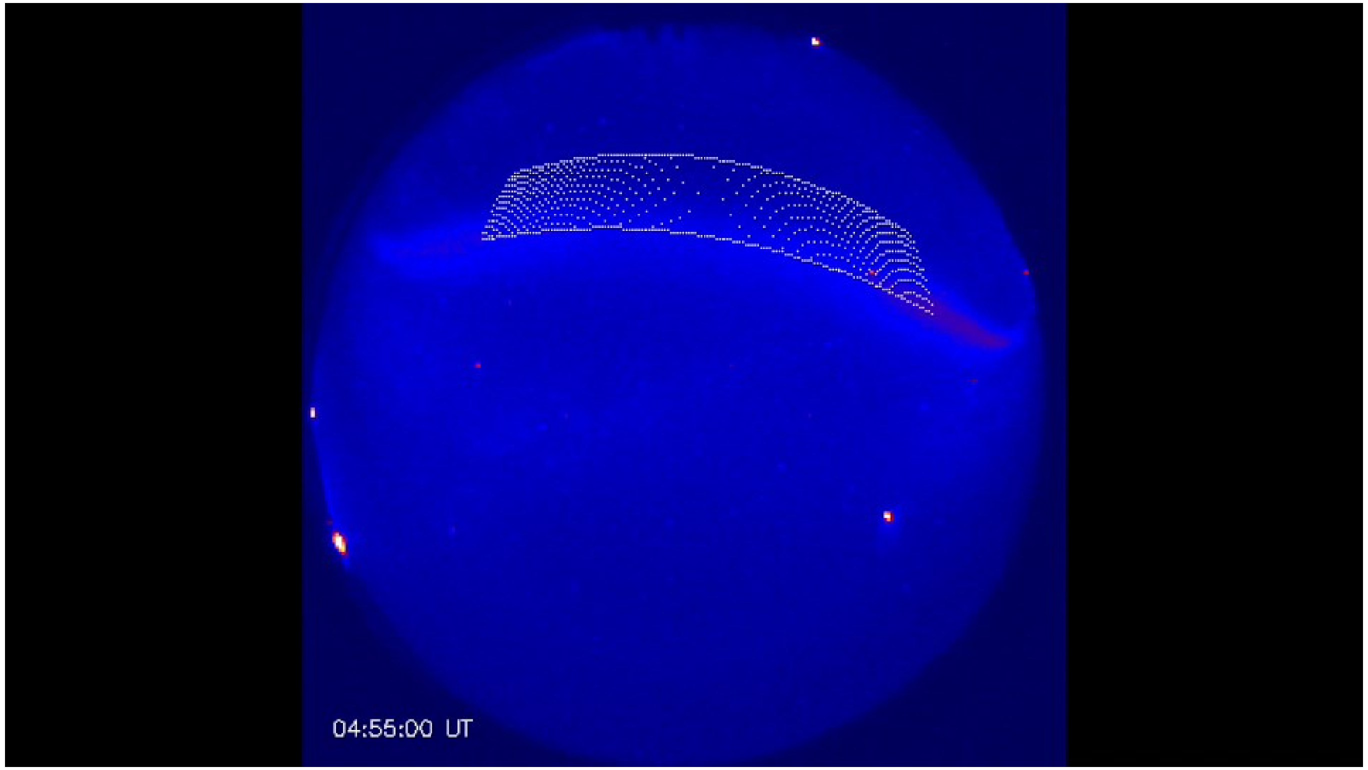
1. Obviously of most importance, this constrains the space of possible eigenvalues.
2. For any marginally unstable mode, there will be singular behavior in the equation.

<u>Proof of 2:</u>

$\frac{\omega_r}{k}$ must be within the range of $v_0(r)$ --> $v_0(r) - \frac{\omega}{k}$ will take on very small values within the radial domain

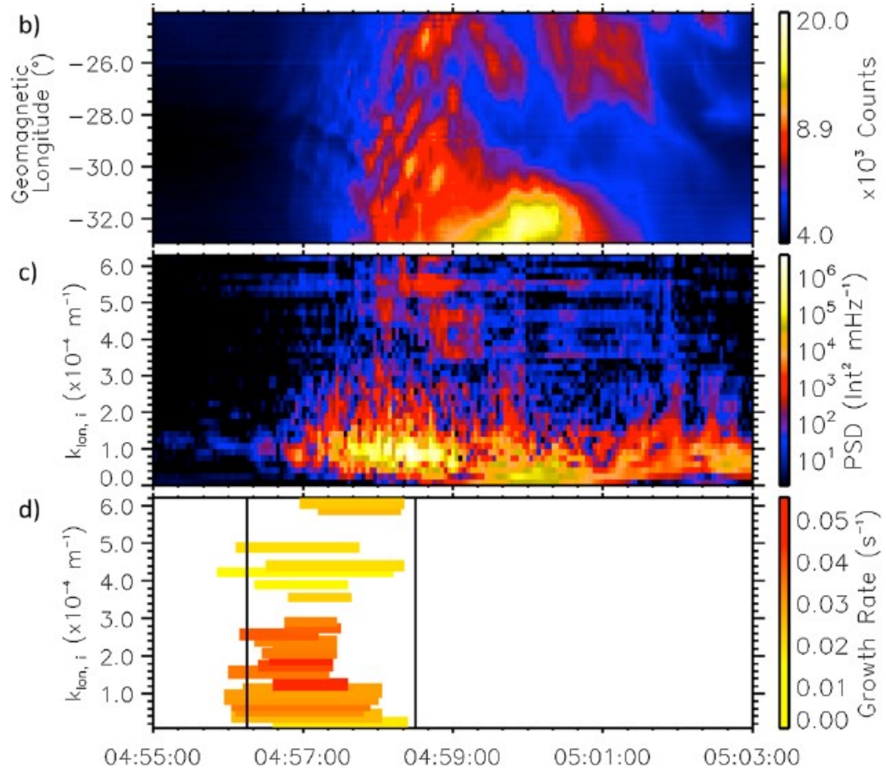
EXTRA SLIDES

Auroral Beads seen by ASI



Optical Analysis

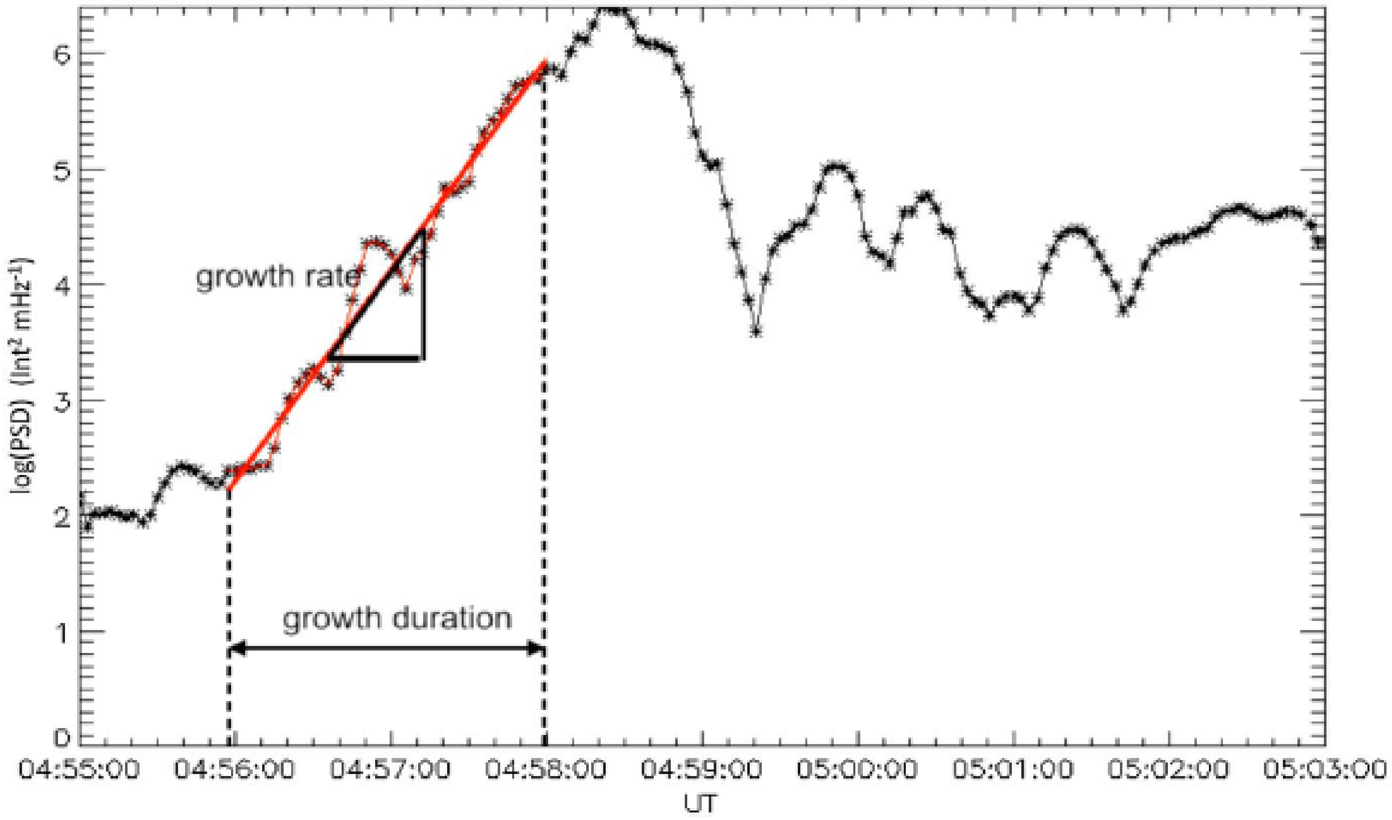
- East-west keogram
- Spatial Fourier transform to obtain spatial periodicity information (intensity → PSD)
- Growth rate for each mode (linear growth assumed)



Growth Rate Determination

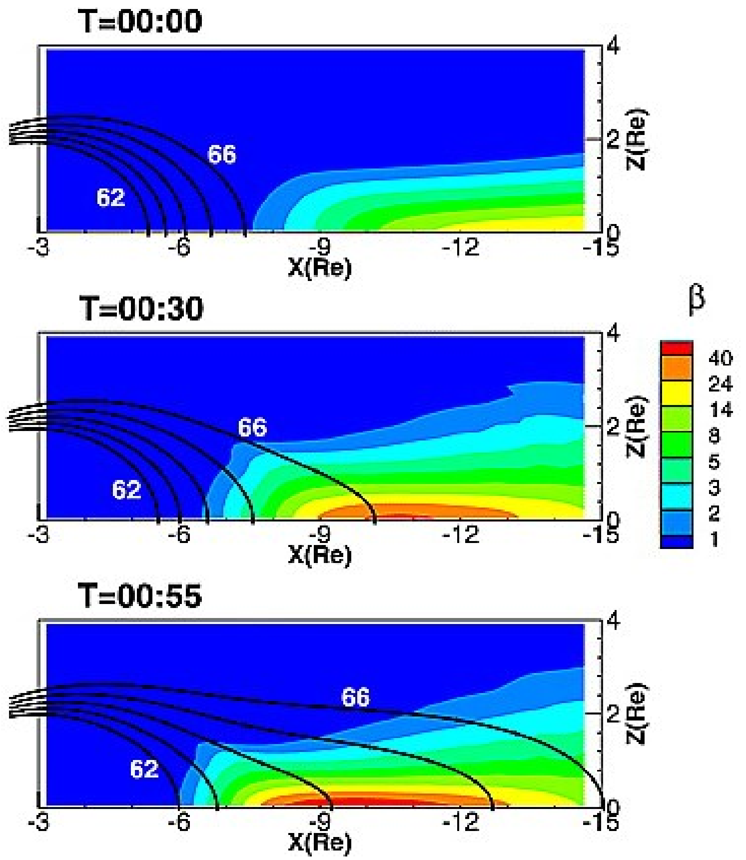
- Example for exponentially growing mode at $k_{lon} = 0.9 * 10^{-4}\ m^{-1}$

$$\frac{d\delta A(t)}{dt} = \gamma^{*}\delta A(t)$$



Field-Line Mapping

- Map magnetotail instability along field lines to ionosphere
- Field lines stretch during growth phase
- Kalmoni (2015) used equilibrium model to enact field-line mapping.
  - Underestimates stretching
- We use dynamical model to obtain a better estimate of relevant spatial scales.



Statistical Analysis of Substorms

- Analysis of 17 substorm events
- Max growth rates range over $[0.03, 0.3]\ s^{-1}$, median $\sim 0.05\ s^{-1}$.
- Ionospheric $k_{lon,i}$ → magnetospheric $k_{lon,m}$ growth rates peak within $k_{lon,m} \in [2.5, 5.0] * 10^{-6}\ m^{-1}$
- Most unstable spatial scales map to $\lambda \approx 1700 - 2500\ km$ in equatorial magnetosphere at $\sim 9 - 12\ R_E$

References

- Derr, J., Horton,W., &Wolf, R. (2019). Shear flow-interchange instability in nightside magnetotail as proposed cause of auroral beads as a signature of substorm onset. J. Geophys. Res. Space Physics.
- Dungey, J.W. (1961). Interplanetary magnetic field and the auroral zones. Physical Review Letters, 6(2), 47-48.
- Kalmoni, N. M. E., Rae, I. J., Watt, C. E. J., Murphy, K. R., Forsyth, C., & Owen, C. J. (2015). Statistical characterization of the growth and spatial scales of the substorm onset arc. J. Geophys. Res. Space Physics, 120, 8503-8516.
- Kepko, L., McPherron, R. L., Amm, O., Apatenkov, S., Baumjohann, W., Birn, J., . . . Sergeev, V. (2015). Substorm current wedge revisited. Space Sci. Rev., 190, 1-46.
- NASA. (2007, January). Themis ground-based all-sky imager (asi) array. https://www.nasa.gov/mission pages/themis/spacecraft/asi.html.
- NASA. (2014, August). The solar wind. https://solarscience.msfc.nasa.gov/SolarWind.shtml.
- Neugebauer, M., Goldstein, B. E., McComas, D. J., Suess, S. T., & Balogh, A. (1995). Ulysses observations of microstreams in the solar wind from coronal holes. J. Geophys. Res., 100, 23389-23395.
- Toffoletto, F., Sazykin, S., Spiro, R., & Wolf, R. (2003). Inner magnetospheric modeling with the rice convection model. Space Sci. Rev., 107, 175-196.
- Wolf, R. A., Tooletto, F. R., Schutza, A. M., & Yang, J. (2018). Buoyancy waves in earth's magnetosphere: Calculations for a 2-d wedge magnetosphere. J. Geophys. Res. Space Physics, 123.

Temporal Modification Determination

- Use amplitude modulus squared.
- Time-average over oscillations since: $\gamma^* \ll \omega^* \implies \frac{2\pi}{\omega^*} \ll \frac{1}{\gamma^*}$
- Now, we arrive at:

$$\frac{d\,|\delta\bar{A}(t)|^2}{dt} = 2\gamma^*\,|\delta A|^2 + C\,|\delta A|^4$$

- Sign of C determinable for any field from the above and:
  - C > 0 → Nonlinear growth (relaxation at higher order or external suppressor effects).
  - C < 0 → Nonlinear relaxation (see following).

Nonlinear Relaxation

- Solve differential equation:

$$\frac{d\,|\delta\bar{A}(t)|^2}{dt} = 2\gamma^*\,|\delta A|^2 + C\,|\delta A|^4$$

- to obtain:

$$|\delta A(t)|^2 = \frac{2\gamma^*}{C}\frac{1}{\frac{v}{C} * e^{-2\gamma^* t} - 1},$$

- Asymptotically, the field amplitudes saturate at:

$$|\delta A|_{sat} = \sqrt{\frac{2\gamma^*}{C}}$$

Nonlinear Analysis (Spatial Modification)

- Substorm must eventually terminate with relaxation of the auroral beads.
- Linear stage:

$$\frac{\partial}{\partial t}x^i = -iM_1^{ij}x^j$$

$$x_1^j = \begin{bmatrix} \delta P \\ \delta B_\phi \\ \delta v_r \\ \delta v_y \end{bmatrix}$$

- Second order terms included:

$$\frac{\partial}{\partial t}x^i = -iM_1^{ij}x^j + M_2^{ijk}x^j x^k$$

- Truncate and solve for second order vector (spatial modification):

$$x_2^i = -i\left(M_1^{-1}\right)^{ij} M_2^{jkl}x_1^k x_1^l$$

Nonlinear Analysis (Temporal Modification)

- Form of all perturbations including nonlinear modification:

$$\delta A(t)e^{-i\omega_r t}(f(y,r) + \Delta(y,r))$$

- Now, we want to substitute into full matrix equation to determine nonlinear time behavior by obtaining time derivative of $\delta A(t)$. It is only exponential growth at linear stage.

Eliminate Spatial Modifications

$$\frac{\partial}{\partial t}\delta A e^{-i\omega_r t}(f+\Delta) = -iM_1\delta A e^{-i\omega_r t}(f+\Delta) + Q$$

$$e^{-i\omega_r t}(f+\Delta)\frac{\partial}{\partial t}\delta A - i\omega_r \delta A e^{-i\omega_r t}(f+\Delta) = -iM_{1H}\delta A e^{-i\omega_r t}(f+\Delta) - iM_{1AH}\delta A e^{-i\omega_r t}(f+\Delta) + Q$$

$$f\frac{\partial}{\partial t}\delta A + \Delta\frac{\partial}{\partial t}\delta A = -iM_{1AH}\delta A(f+\Delta) + Qe^{i\omega_r t}$$

$$\frac{\partial}{\partial t}\delta A = -if^* M_{1AH}\delta A f + f^* Q e^{i\omega_r t}$$

$$\frac{\partial}{\partial t}\delta A = \gamma\delta A + f^* e^{i\omega_r t} Q$$

- 1. Chain Rule (L), Matrix Split (R)
- 2. Multiply by $e^{i\omega_r t}$, Hermitian cancellation
- 3. Multiply by f^*, Anti-Hermitian cancellation
- 4. Simplify Anti-Hermitian term